Lines through points

Here is something to try. Place five distinct points on a plain piece of paper in such a way that any line that goes through two of them must pass through a third. Do the same with six points. One way to do this is to make all the points lie on the same straight line. Is this the only way to do it? What happens if you have four points? or more than six points?

Put another way, suppose that you put a finite number of points on a page in such a way that they do not all lie on the same line. Then, is it necessarily true that you can find a line that passes through exactly two of the points? This is obvious if there are two points, so suppose that there are at least three.

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The answer to this question is **yes**. Suppose we have finitely many points not all on a straight line. Then, given a point P, we can find a line through two other points that does not contain P. (To see this, let Q be a second point. Since we are assuming that PQ does not contain every point of the set, there is a point R that does not lie on PQ. But then P cannot lie on QR.)

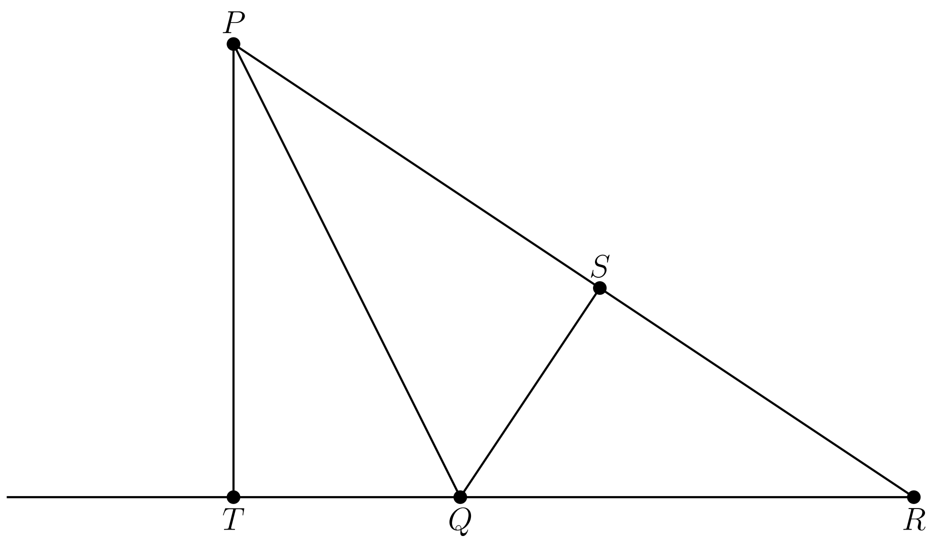
To get a feel for the situation, suppose we have six points not all in a line. (It would help to make a diagram.) For one of the points A, we can find a line BC through two other points that does not contain A. If B and C are the only points on this line, then we are done. Otherwise, there is a third point D on the line BC and we can see that at least one of the lines AB, AC and AD contains only two points of the set (since there are only six points).

If you increase the number of points, then this sort of argument becomes complicated and unmanageable. In fact, the problem was unsolved for quite a while. Eventually, the following elegant argument was found.

For each point P, we measure the distance from P to each of the lines through pairs of points that do not contain P. (The distance from a point to a line is the length of the perpendicular from the points to the line; think of the height of a pole as the distance from its top to the ground.) Since there are finitely many points and finitely many lines, we can find a point P and a line not containing P for which this distance is positive and minimal, no larger than the distance for any other point-line pair.

Now we argue by contradiction. Assume that the result is false. That is, we assume that any line through two of the points must contain a third point and obtain a consequence that is clearly false. In particular, we show that this assumption forces us to conclude that we can find a point-line pair for which the distance is smaller than for the point P. This conflicts with our choice of P and its corresponding line as a minimizing pair. Thus, we must abandon our assumption that the result is false.

Draw a line though P perpendicular to its corresponding line that meets it at T. The length of PT is the distance from P to the line. This diagram illustrates the situation.



Since each line through two points contains a third, there must be two points, Q and R, of the set on the same side of T, with Q between T and R. Draw the line PR and consider the distance from Q to PR. This is the length of the seqment QS, where S lies on PR and QS is perpendicular to PR. Then the distance between Q and the line PR is strictly less than the distance from P to the line QR. This goes against our assumption that the minimum distance is realized by P and its corresponding line.

(To see that QS is less than PT, we note that the angles of the two right triangles PTR and QSR are correspondingly equal, so that triangle PTR is a scaled up version of triangle QSR. Therefore the side PT of the first triangle is longer than the corresponding side QS of the second.)