The problem of the four points

Get a clean sheet of paper and place four dots on the page. There are six pairs of dots; take a ruler and measure the distance between any pair of them. In general, the six measurements will all be different. However, it can certainly occur that the distance between one of the pairs of dots is equal to the distance between another pair of dots.

A natural question to ask is whether you can place the dots so that the distance between any pair of dots is the same as the distance between any other pair. If the dots are placed on a flat surface, this cannot be done. If you are allowed to go into the third dimension, then you can have the dots at the vertices of a regular tetrahedron (or a pyramid whose base is an equilateral triangle and whose height is arranged so that all four faces are identical in size and shape).

So we can ask, rather, whether we can set the dots so that the six distances between pairs of dots have exaactly two distinct values. At this point, I will pull a starry curtain across the column, so that you can think about it before reading further.

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When people are given this problem, often the first configuration that comes to mind is a square. If the four dots are at the vertices of a square, then the four sides of the square have one length and the diagonals another.

One strategy to get a suitable confinguration is to start with three dots and ask whether we can make the distance between any pair of them the same. The answer is, yes; simply let them be the vertices of an equilateral triangle. Then it is a matter of introducing a fourth point in such a way that you introduce no more than one new distance.

However, there may be configurations which do not involve an equilateral triangle; the square is one. Can you find any others? How many distinctly different solutions to the problem can you find?