A game for two

Here is a game you can try with a friend. There are two players and they make their moves alternately. The first player picks any whole number from 1 to 9, inclusive. The second player then picks a different number. The game continues, with each player in turn picking a number that has not already selected. So the game can go on for no more than nine moves.

What is the goal? Each player tries to pick numbers in such a way that three among them add up to 15. The first player that does so is the winner. If at the end when all the numbers have been picked, neither player can find three among his selection adding to 15, then the game ends in a draw.

Suppose the first player, Ann, chooses the number 8. Let the second player, Bob, pick a 4. Ann then selects 2. Bob notices that Ann already has 8 and 2, so for his second choice, he should grab the number 5. What should Ann do now? To frustrate the designs of Bob, she picks 6. Bob picks 7 and then Ann picks 1. Ann is the winner because three of her numbers are 8, 6 and 1, which add to 15.

Try this game for a while. Is there any way that either player can force a win, regardless of what the other does? Or can both players make sure that the game ends in a draw?

It is possible to see that the seoond player cannot force a win. Since the argument is a little subtle, I will put it in parentheses, so you can skip it if you want. (Suppose, if possible, that Bob has a winning strategy. Then Bob can win, whatever number Ann chooses to begin with. So suppose, for sake of argument, Ann picks a 9. Let us start a second game, where Bob is the first player, but let Bob pretend that Ann has already played 9 and pursue his winning strategy. If Ann somewhere along the line in fact plays a 9, this does not affect Bob's strategy. If Ann does not play a 9, Bob's strategy will still win. So in this new game with Bob going first, Bob will guarantee that he wins. But then this means that the first player has a winning strategy. This is impossible, since both players cannot each force a win. So the second player does not have a winning strategy, and the best he can do is force a draw.)

I have found that most people are not familiar with this game, so they are at a bit of a loss as to how to proceed. But through the magic of mathematics, we can make it a little more familiar. In a past column, I introduced you to the magic square:

4	9	2
3	5	7
8	1	6

In this array of numbers, three numbers add up to 15 if and only if they belong to the same row, the same column or the same diagonal. To play the game, Ann and Bob both visualize a 3×3 square grid. When Ann picks a number, she puts **X** in the cell of the grid that corresponds to the position of the number in the magic square. When Bob picks a number, he puts **O** in the cell of the grid that corresponds to the position of the number in the magic square. Thus, Ann begins with **X** in the lower left corner, and Bob follows with **O** in the upper right corner. After each has performed two moves, the grid looks like

O		**X**
	O	
X		

Notice than Bob's second choice corresponds to the middle of the grid and prevents Ann from completing a diagonal. However, if Bob were allowed to choose a 6 next, then Bob would have three numbers adding to 15 and complete a diagonal. So Ann has to grab 6 at her next turn, thus presenting the situation

O		**X**
	O	
X		**X**

You may have realized by now is that we are really playing noughts-and-crosses, a game that many of you will be familiar and comfortable with. We have here an example of a mathematical *isomorphism* between two structures that are superficially different but mathematically identical. In our two games, each move in one game corresponds to a move in the other. The winning condition in one game – find three numbers adding to 15 – corresponds to the winning condition in the other – put three of your symbols in the same row, column or diagonal. Everything we know about one game can be applied to the other. We have in effect taken something that looks strange and difficult to understand and dressed it in clothes that make it familiar and easy to deal with.

This is a fundamental idea, one which underlies applications in mathematics. We take some situation in the real world, and try to formulate a mathematical structure that mirrors it. We can formulate real world conditions in mathematical terms, manipulate the mathematics, and then read the mathematical results back into the real situation.