The Ratchet

Look at this table of the powers or 2 and 5;

n	2^n	5^n
1	2	5
2	4	25
3	8	125
4	16	625
5	32	3125
6	64	15625
7	128	78125
8	256	390625

As the exponent n grows, the corresponding powers of 2 and 5 grow larger and have more digits. However, these powers are deferential towards each other. Every time n increases by 1, exactly one of the powers gains one more digit while the other remains the same length. For example, 2^4 has one more digit than 2^3, while 5^3 and 5^4 have the same length. If we continue computing powers, will this go on forever?

To get a handle on how we can find out, let us look at the particular case $n = 5$ and see what it means for $2^5 = 32$ to have two digits and $5^5 = 3125$ to have four digits. This really says that $10 < 2^5 < 10^2$ and $10^3 = 1000 < 3125 < 10^4 = 10000$. Taking these two inequalities together, we can check that $10^4 < 2^5 \times 5^5 < 10^6$.

Now let us look at a general number n, and suppose that 2^n has x digits and 5^n has y digits. Then $10^{x-1} < 2^n < 10^x$ and $10^{y-1} < 5^n < 10^y$. Taking these two inequalities together, we find that

$$10^{x+y-2} < 2^n \times 5^n = 10^n < 10^{x+y}.$$

Since n lies between the whole numbers $x + y - 2$ and $x + y$, we must have that $n = x + y - 1$. In other words, the number of digits in 2^n and the number of digits in 5^n must add up to $n + 1$.

The consequence of this is that every time we increase the exponent by 1, we increase the total number of digits in the corresponding powers of 2 and 5 by one. This can happen if and only if one of the powers gets one more digit and the other stays the same length.

If you are a secondary mathematics student and want a little bit more of a challenge, consider the following. Suppose we start by writing the powers of 10 to base 2: $10 = (1010)_2 = 1 \times 2^3 + 1 \times 2$; $100 = (1100100)_2 = 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^2$; $1000 = (1111101000)_2$, and so on. Now write the powers of 10 to base 5: $10 = (20)_5 = 2 \times 5$; $100 = (400)_5 = 4 \times 5^2$; $1000 = (13000)_5 = 1 \times 5^4 + 3 \times 5^3$; $10000 = (310000)_5 = 3 \times 5^5 + 1 \times 5^4$, and so on. Then for any whole number k greater than 1, there is exactly one power of 10 than has k digits in either base 2 or in base 5 (but not in both bases). I would be interested to see what some reader does towards proving this fact.