## A magic squares

According to legend, a turtle climbed out of an ancient Chinese river. In its shell was inscribed a $3 \times 3$ square array of numbers, the digits from 1 to 9 inclusive, each appearing once in such a way that the three numbers in each row, in each column and in each of the two diagonals, all had the same sum. This is the original "magic square", and before reading on, you might wish to reproduce it on your own.

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While a lot of people might arrive at an answer through trial and error, it is possible to shorten the process using a little systematic analysis.

What must the common sum along the rows, columns and diagonals be? Since all the digits add up to 45 and there are three rows, the sum along each row must be 15. A good next step is to see how you can add three digits together and get 15. The possibilities are

$$15 = 9 + 5 + 1 = 9 + 4 + 2 = 8 + 6 + 1 = 8 + 5 + 2 = 8 + 4 + 3 = 7 + 6 + 2 = 7 + 5 + 3 = 6 + 5 + 4.$$

There are exactly eight possibilities. We have to accomodate eight sums in the square array, since there are three rows, three columns and two diagonals. So each of the eight sums is realized somewhere in the diagram. The number that goes in the middle cell must figure in four of the sums. Each corner digit is in three of the sums, and each mid-edge digit in two.

Up to a rotation and reflection of the figure, there is one answer:

| | | |
|---|---|---|
| 4 | 3 | 8 |
| 9 | 5 | 1 |
| 2 | 7 | 6 |

There are a few more magic characteristics of this array. The products of the numbers in the three rows are 96, 45 and 84; added together, they give 225 which is the square of 15, the sum of each row. Similarly, the products of the numbers in the three columns are 72, 105 and 48; their sum is again the number 225.

There is more. Using the digits in the rows to create three three-digit numbers gives (438, 951, 276). Writing these numbers backwards gives (834, 159, 672). Not only two triples of three numbers have the same sum, their squares also add to the same thing:

$$438^2 + 951^2 + 276^2 = 834^2 + 159^2 + 672^2.$$

We can do a similar thing with the columns and find that the two triples (492, 357, 816) and (294, 753, 618) have the same sum and square-sum. If we go in the diagonal directions, we can two more pairs of triples with the same property:

$$(456, 231, 978), (654, 132, 879)$$

and

$$(852, 174, 639), (258, 471, 936).$$

We conclude with a paper-and-pencil game played by two people moving alternately. The first player picks a digit between 1 and 9. After that, each player picks a digit not already selected so far. The winner is the first person to find among the numbers she has selected, *three* that add up to 15. These three numbers need not be consecutive choices by the player. The game ends in a draw when all nine digits have been chosen and neither player has three summing to 15. Try this game out with a friend. I will leave it to the reader to see if there is any relevance to the magic square.