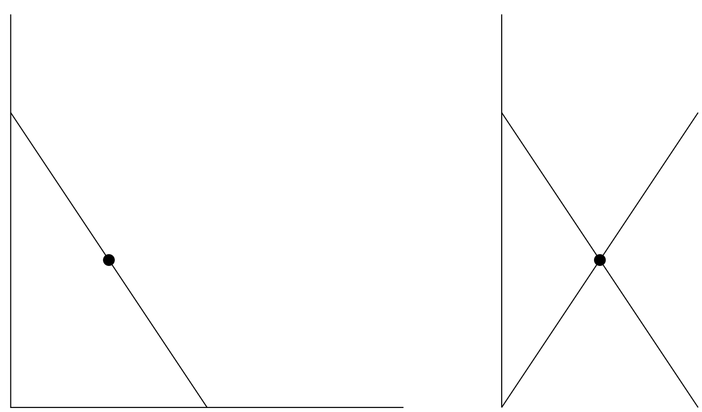
Geometry problems for elementary and middle school

G.1. Ladders and ironing board.

Suppose that a ladder is leanding against the wall of a house; the wall and the ground are both completely flat and perpendicular to each other. The ladder begins to slide, its ends maintaining contact with the wall and ground. What is the path traced out by the midpoint of the ladder.

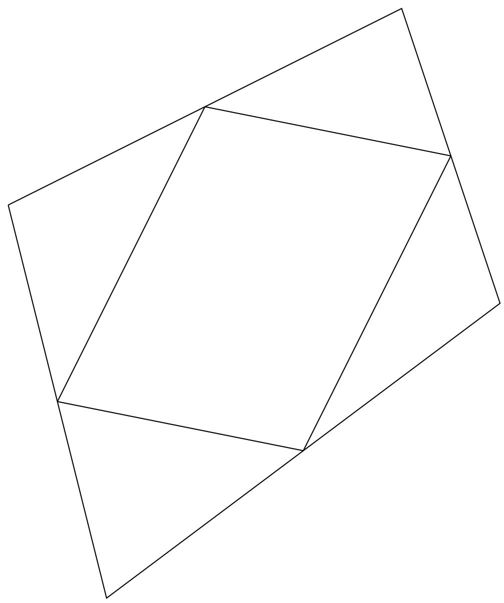


Another situation which exhibits the same phenomenon is that of a collapsible ironing board, that has a pair of legs crossed at their midpoints. The end of one leg is fixed to the end of the ironing board; the end of the other is free to travel along the ironing board. What is the path traced out by the crossing point when the ironing board is opened.

G.2. A quadrilateral within a quadrilateral.

Draw any quadrilateral (this is a polygon with four sides). Find the midpoints of the four sides and join adjacent pairs of them to make another quadrilateral inside of the first. What special thing can you say about this inner quadrilateral?

The original quadrilateral is partitioned into five regions by the lines joining the midpoints of the sides. Can you see (without doing any computation) why the area of the inner quadrilateral is equal to the sum of the areas of the four triangular "ears"?

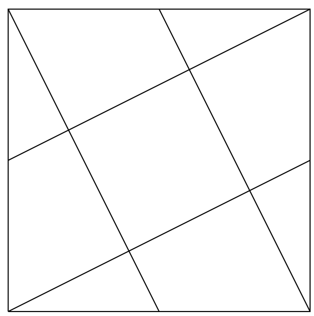


3. Partitioning a square.

Show how to cut a square into three pieces in such a way that the pieces are similar; this means that each piece is a scales version of another. This is easy to do if you allow two of the three pieces to be the same size, and you can look at situations where each piece is a triangle, or a quadrilateral, or a pentagon. Can the partition be made where all the pieces are different sizes?

4. The square in the middle.

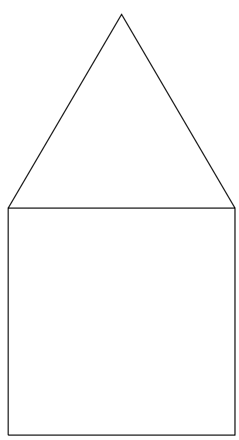
The diagram shows a square with four lines joining a vertex to the midpoint of a side. What is the area of the square in the middle?



When this was given to one class, one student wondered how we knew that the figure in the middle was a square. How would you convince the student that this was so?

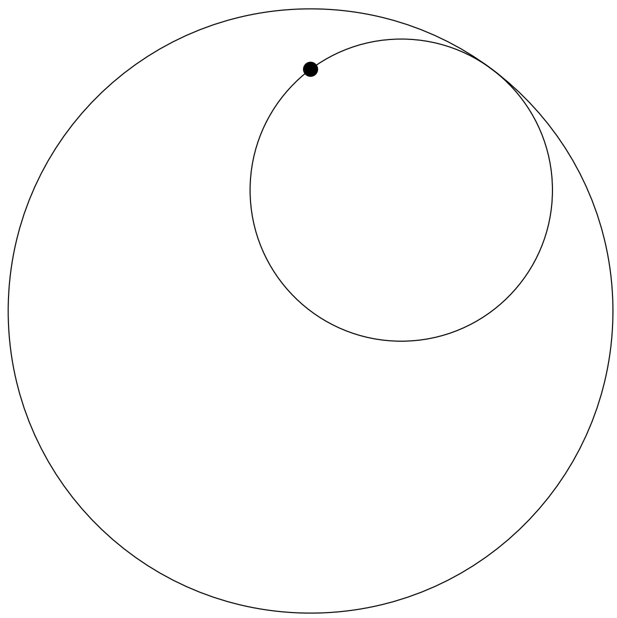
5. A triangle atop a square.

A square of side length 1 shares an edge with an equilateral triangle of the same side length which is external to the square. There is a circle that passes through the two vertices of the square that are not on the triangle and the vertex of the triangle that is not on the square. What is the radius of this circle?



6. A circle rolling inside another circle.

A circle of radius 1 rolls without slipping along the inner side of the circumference of a circle of radius 2. What is the path tranced out by a fixed point on the inner circle?



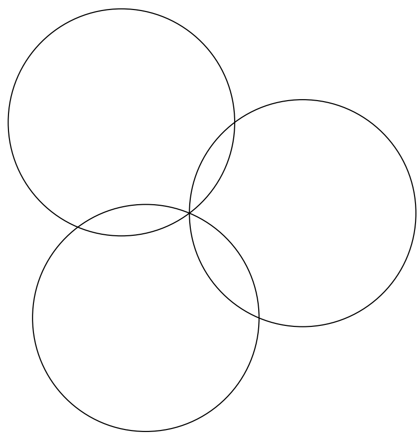
You can experiment at home with your own cutout models. You can slso use take-up reels from an old movie projector, one having double the radius of the other. The curve is a particular example of a *hypocycloid*, and recourse to the definition of this curve of Wolfram's website will give you a dynamic representation of the path. (For some reason, Wikipedia does not give this special case.)

7. The four points.

Put four points on a page. There are six pairs of points; measure the distance between each pair. Generally, you will get six different values. Is it possible to have a configuration where all the values are the same? Determine all configurations where there are exactly two distinct values.

8. The beer mug problem.

Suppose that we have a mug of cold beer, condensation eveloping the outside. We place it on the counter, and it leaves a watery circular mark. We take a sip, and place it again on the counter, so that the second circular mark intersects the first. We take another sip, and place it on the counter a third time so that the third circular mark passes through one of the points common to the first two marks. It is now possible to have another sip and place the mug on the counter so that the fourth circular mark passes through the three points that are common to exactly two of the previous circles. Why is this?



Note: You can get some insight into a solution to this problem by drawing in all the line segments that are known to be equal in length to the common radius of the three circle; each centre of the first three circles is joined to the three points where it intersects the other two circles. The result is what looks like part of a two-dimensional representation of a cube, showing nine of the twelve edges. Sketch in the other three edges.

9. Gifts from pigeons.

On the ground in a garden is a large marble slab in the form of an equilateral triangle of side length two metres. A flock of pigeons fly over, dropping poop at five separate locations on the slab. Prove that two of these spots are no more than one metre apart.

Note: This problem can be solved by a gambit known as the Pigeonhole Principle, which simply says that if you have to sort a set of objects into categories and there are more objects than categories, then there will be some category with at least two objects. Applying this principle, one can state with certainty that at the Last Supper, there were two attendees born in the same month. Here, the trick is to think of the slab as made up of four equilateral triangles with side length one metre. Another application of this principle is Puzzle 9.