**Problems on Combinatorics**

1. Miss Dawe gets on a Bathurst streetcar at the Bloor subway station and rides it to the other end of the line at the Exhibition. The whole journey requires 24 minutes, and every three minutes on the journey her streetcar passes one going in the opposite direction. What can be said about the total number of streetcars on the line? If a person arriving at a stop has just missed a streetcar, how long can she expect to wait for the next one?

2. Give a combinatorial argument for each of the following equations:

   (a)
   $$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} .$$

   (b)
   $$\sum_{r=0}^{n} \binom{n}{r} = 2^n .$$

   (c)
   $$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n} .$$

3. There are ten items on sale at a bazaar, each costing less than one dollar. Prove that it is possible for two people to purchase distinct subsets of these objects and pay exactly the same amount. (Not all the objects need to be purchased.)

4. Each of a group of ten individuals sends cards to exactly five others of them. Prove that there is always a pair of individuals for which each sent a card to the other.

5. You have a set of twelve billiard balls, each identical in appearance. Eleven of them weight the same, and the twelfth has a different weight. Using no more that three applications of an equal-arms balance, explain how to determine which is the ball of different weight and whether it is heavier or lighter than the other eleven.

6. You have five balls, no two weighing the same. Using an equal-arms balance, in a minimum number of comparisons, show how to arrange them in increasing order of weight.

7. How many ways can seven letters be distributed into three boxes in each of the following situations?

   (a) The letters and the boxes are distinguishable.

   (b) The letters are distinguishable, but the boxes are not.

   (c) The boxes are distinguishable, but the letters are not.

   (d) Neither the letters nor the boxes are distinguishable.

8. The street plan of a village is a square $3 \times 3$ grid of nine blocks, each with a side length of 100 metres and each surrounded by a road, so that there are 24 100-metre segments of road separating adjacent blocks and along the boundary. A boy starts in the corner of the village and rides his bike so as to cover each 100-metre segment at least once. What is the shortest possible journey?

9. There are 2000 apples contained in several baskets. You are allowed to remove baskets and/or remove any number of apples from the baskets. Prove that you can do this in such a way that there are at least 100 apples, with the same number of apples in each of the remaining baskets.

10. Each of the students in a class writes a different two-digit number on the blackboard. The teacher claims that, no matter how this is done, there will be at least three numbers on the board the sum of whose digits are the same. What is the smallest number possible in the class for the teacher to be correct?

11. Five swimmers leave their shoes on the beach. When they get them later, each swimmer takes one left and one right shoe at random. In how many ways can this be done so that there is no matching pair?

12. In how many ways can five couples be paired off so that two husbands are paired with their own wives and the other three are not?

13. A $7 \times 7$ square is made up of 16 $1 \times 3$ tiles and one $1 \times 1$ tile. Prove that the $1 \times 1$ tiles is either at the centre of the square or adjoins one of the boundaries.

14. For each of the powers of 10 from 10 to 100,000, determine the following two numbers: (1) the number of digits required to write down all the numbers from 1 up to that power, inclusive, and (2) the number of zeros that occur among the digits of the numbers from 1 to that power, inclusive. For example, for 10 itself, it requires 11 digits to write the numbers from 1 to 10, , inclusive, and among these digits there occurs exactly one 0.

    What do you observe? Make a conjecture and try to prove it.

15. You are given a sufficiently large supply of coppers, nickels and dimes. How many distinct amounts of money can you make up from

    (a) Exactly three coins?

    (b) Exactly six coins?

16. Suppose that you are given an $8 \times 8$ standard chess-board. A coin sits in the lower left corner. A path is a sequence of moves; each move is from one square to an adjacent square, where one moves either upwards or to the right. For each square of the chess-board, enter into that square the number of paths from the lower left corner to the square. For example, the lower left corner in the first row from the bottom and the first column from the left will have the number 1 entered into it, as there is one path (just leave the coin where it is). There are ten possible paths to get the coin from the lower left corner to the square located in the fourth row from the bottom and the third column from the left, so in this square you will enter the number 10.

17. In the game of *noughts-and-crosses* (also known as *tictactoe* or *X's-and-O's*), two players play alternately into a $3 \times 3$ array of unit squares, the first player marking his chosen square with an $X$ and the second with an $O$; no player can play into a square that has already been taken. The winner is the first player to have placed his symbol into the three squares of any row, column or diagonal or the array.

    The first player has essentially three distinct choices of move: play into the centre square; play into a corner square; play into a square in the middle of an edge.

    (a) List all the essentially different configurations after each player has made one move (so that there is one $X$ and one $O$ written into the array). (You will have to decide what *essentially different* means.)

    (b) List all the essentially different configurations after three moves have occurred, two by the first player and one by the second (so that the array now contains two $X$'s and one $O$). Classify the configurations according as (i) the first player can force a win regardless of what moves his opponent makes; (ii) the second player can force a win; (iii) both players can force a draw.

18. (a) In a pair of *fair* dice, for each die, the six faces are numbered from 1 to 6 inclusive, and there is an equal chance that each individual face will appear. For the total of the two dice, there are eleven possibilities, but they are no all equally probable, since some like 7 can be obtained with many individual configurations of the two dice, while others like 11 (which can only be obtained if the two die show 6 and 5 in some order) can come from very few configurations of the two dice. Determine the probability of each outcome for a random toss of two fair dice, and explain why your answer is correct.

    (b) It has been known for some unscrupulous gamblers to *load* the dice; this means that they skew the weight distribution of each die so that some faces are more likely to show than others. Is it possible to load a pair of dice each marked in the usual way in such a way that each of the outcomes from 2 to 12 inclusive are equally likely?

    (c) Expand the following square of a polynomial:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 .$$

    Explain why for each power $x^k$ in the expansion, the coefficient is equal to the number of ways that $k$ can be attained with two ordinary dice.

    (d) Is it possible to take a pair of fair dice and renumber the faces with positive integers in a way other than the usual fashion (with numbers from 1 to 6 inclusive) so that, for each of the numbers from 2 to 12 inclusive, the number of ways that number can be obtained with a roll of the two dice is the same as the number of ways when the dice are numbered in the standard way?

    (*Comment.* I notice that in recent educational writings that, presumably out of deference to those whose religious views do not countenance gambling, *dice* are referred to as *number cubes*. Those of you who have scruples on this point are welcome to use the revised terminology.)

19. An old-fashioned toaster can toast two slices of bread at the same time, but only on one side. Each side has a "door", and in a well-designed toaster of this type, when the doors are opened, the slices of bread automatically turn over and you can close the door and toast the opposite side. (Note that the elements are always " on" while the plug is in; thus, it was traditional for young children to be assigned the chore of "watching the toast" to make sure that it did not burn.)

    Suppose that you want to toast three slices of bread on both sides, and it takes one minute to toast one side of a slice. What is the minimum amount of time to do the job?

20. Let $n$ be a positive integer. Imagine that we have a large supply of numerals 1, 2, $\cdots$, 9 and that we wish to label a line of telephone poles from 1 up to $n$. Let $f(n)$ be the number of 1's that are required for the job. For example, $f(9) = 1$, $f(10) = 2$, $f(11) = 4$ and $f(20) = 12$.

    (a) Determine $f(99)$, $f(100)$, $f(200)$, $f(365)$, $f(1000)$ and $f(100,000)$.

    (b) Is it possible for $f(n)$ to be actually larger than $n$? Is it possible for $f(n)$ to be equal to $n$? If your answer to either question is "yes", give examples. If "no", explain why.

    (c) Is this statement true or false: If $n$ is large enough, then $f(n)$ is always larger than $n$? Explain.

21. A $5 \times 9$ rectangle is partitioned into a set of 10 rectangles, each of whose dimensions is a positive integer. Prove that some two of the ten rectangles must be congruent.

22. The five interlocking Olympic rings are coloured black, blue, red, green, yellow, but not necessarily in this order. The members of the class are asked to colour the rings. However, as they are not given the order of the colours, each one colours at random, using the five colours.

(a) How many different ways are there of colouring the rings?

(b) What is the probability that there are two people who have coloured the rings exactly the same way?

(c) Look up the correct colouring. What is the probability that someone in the class will have found it?

What is the probability that at least one colour will be chosen correctly for a random colouring?

23. Consider the numbers 1, 2, 3, 4. There are twelve possible ways of finding an ordered pair of these numbers: (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3). Let us form the four-by-four matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

You will observe that every possible ordered pair appears as an adjacent pair in some row of the matrix and also in some column of the matrix.

Find a five-by-five matrix for which every one of the twenty ordered pairs from the numbers 1, 2, 3, 4, 5 appears as adjacent elements in some row. Can you also arrange at the same time that every ordered pair also appears as adjacent elements in some column? Answer the same question for a six-by-six matrix exhibiting pairs of numbers 1, 2, 3, 4, 5, 6.

24. A pile of 25 poker chips is split into two piles. One of the piles containing more than one chip is split into two piles, so that there are now three piles. This continues for a number of moves, at each of which a pile with at least two chips is selected and split into two piles. Eventually, the process terminates with 25 piles, each with one chip.

What is the minimum and the maximum number of moves required before the process terminates. At each move, the numbers of chips in the two piles obtained from a single pile are multiplied together. What is the sum of the products so found over all of the moves?

For example, starting with five chips, we might have the following sequence of pile numbers: (5), (3,2), (2,1,2), (1,1,1,2), (1,1,1,1,1), with the products 6,2,1,1 that add up to 10.

25. Let $n$ be a positive integer. Prove that the number of $2n-$digit numbers, with $n$ digits equal to 1 and $n$ equal to 2, is equal to the number of $n-$digit numbers whose digits are all 1, 2, 3 or 4 and for which the number of 1s is equal to the number of 2s.

For example, when $n = 3$, we have the $6-$digit numbers 111222, 112122, 121122, 211122, 112212. 121212. 211212. 122112, 212112, 221112, 112221, 121221, 211221, 122121. 212121. 221121, 122211, 212211, 221211, 222111 and the $3-$digit numbers 123, 124, 213, 214, 132, 142, 231, 241, 312, 321, 312, 421, 333, 334, 343, 433, 344, 434, 443, 444.

26. On the Island of Camelot, there live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they immediately both change to the third colour. Is it possible that they will all eventually be the same colour?