Case study: The area under a cycloid

In the sixteenth and seventeenth century, before the advent of calculus, many problems in quadrature were solved using *ad hoc* methods. Such methods were depended on particular attributes of the situation, whereas the approach of calculus depended on recognizing the problems as belong to broader categories. In either case, the method of solving the problem teaches us about the situation the goes beyond the mere statement of the result.

A nice example of this is the determination of the area under one arch of a cycloid, the locus of a point on the circumference of a circle that is made to roll without slipping along a line. In particular, this area is equal to three times the area of the generating circle.

1. A pre-calculus determination. We will determine the area under half the arch and suppose that P is a point on a circle the rolls through half a revolution along a line. Initially, P is coincident with the point A on the line; after a half revolution, P is at position C, whose distance from the line is equal to the diameter of the circle. Let B be a second point on the line, and D a final point, so that $ABCD$ is a rectangle. The length of the side AB of the rectangle is equal to half the circumference of the generating circle $\frac{1}{2}c$, while with length of the side BC is equal to the diatmeter d of the generating circle. Both AB and CD are (parallel) tangents to the generating circle.

Freeze the generating circle in position at a particular moment. Suppose that it touches AB at X and CD at Y. Then $\text{arc}(PX) = AX$. Draw a chord of the circle through P that is parallel to AB; let it meet the circle again at Q. We have that

$$\text{arc}(QY) = \text{arc}(PY) = \frac{1}{2}c - \text{arc}(PX) = \frac{1}{2}c - AX = BX = CY \ .$$

Since $\text{arc}(QY) = CY$, we can think of the position of the generating circle as being the same as the position of a second generating circle, this time rolling along the line CD, with Q intially coinciding with C. Then Q is a position on a second cyloid, similar to the first, that passes from C to A. Indeed, a rotation about the centre of the rectangle $ABCD$ through an angle of $180°$ interchanges the two cycloids.

Thus, we can partition the rectangle $ABCD$ into three regions: (1) a lens-shaped region lying between the two cycloids with area α; (2) the region bounded by the segments AB and BC that lies underneath the lower cycloid with area β; (3) the region bounded by the segments AD and DC that les above the upper cycloid, congruent to region (2) and having area β. The area of the rectangle is

$$\frac{1}{2}cd = \alpha + 2\beta \ .$$

We wish to determine the area $\alpha + \beta$ lying under the half arch of the upper cycloid.

To do so, we find α. A typical chord parallel to AB of the lens-shaped region is equal in length to the chord of the generating circle at the same height. Appealing to Cavalieri's Principle, we see that the lens-shaped region and the circle have equal area, so that $\alpha = \frac{1}{4}cd$. Hence

$$2\alpha = \alpha + 2/beta \Longrightarrow \beta = \frac{1}{2}\alpha \Longrightarrow \alpha + \beta = \frac{3}{2}\alpha \ ,$$

so that the area under the half-arch is 3/2 the area of the circle and the area under the full arch is 3 times the area of the circle.

2. Calculus. If the situation is represented parametrically, then the determination of the area under an arch is an extremely standard exercise. Suppose a circle of radius r rolls along the $x-$axis, starting at the origin, where the generated point P on the cycloid is originally located. After the circle has rolled through an angle θ clockwise, its centre is at the point $(r\theta, r)$ and the generating point P is located at

$$(r\theta - r\sin\theta, r - r\cos\theta) = (r(\theta - \sin\theta), r(1 - \cos\theta)) \ .$$

The area under the cycloid ($\int ydx$) is given by

$$\begin{aligned}\int_0^{2\pi} r^2(1-\cos\theta)d(\theta - \sin\theta) &= r^2\int_0^{2\pi}(1-\cos\theta)^2 d\theta \\ &= r^2\int_)^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta = 3\pi r^2 \ .\end{aligned}$$

The first argument provides us with some indications of the intimacy between a cycloid and its generating semi-circle and makes the final result natural and inevitable. In the second, the particular characteristics of the cycloid are used in setting up the situation, and disappear while the technicalities take over. However, the dynamic character of the locus is seen in our choice of parametrization rather than use the cartesian equation for the cycloid, which would be more awkward and difficult to obtain and a nightmare to deal with in an integration. Even when a proof might be according to standard operating procedures, the necessity of setting it up forces upon us a choice that teaches us more about the situation than is embedded in the statement of the result.

Indeed, this may even apply for computer generated proofs. A computer is not a black box; it has to be programmed, and the programmer must make choices, as indeed must the user of the program. Even here, the proof forces upon us a process of reflection and discovery not present in the result.