Learning Modern Algebra: From early attempts to prove Fermat's Last Theorem
by Al Cuoco and Joseph J. Rotman
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Beyond the quadratic formula
by Ron Irving
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The modern high school syllabus covers very little of what used to be called theory of equations. Students have little opportunity gain fluency with polynomials and appreciate their rich mathematical structure and history. Both these books redress this. The authors have the dual goal of providing teachers with a solid foundation of algebra that might inform their teaching and guidance of more able students, and of providing students and teachers material suitable for self-study. They succeed admirably, albeit in different ways. The organizing principle behind Irving's book is the solution of polynomial equations while that behind that of Cuoco and Rotman is providing some of the apparatus needed to handle the Fermat equation.

While Irving clearly sets out his propositions, he avoids formal proofs and invites the reader to a hands-on understanding by working through examples and sequences of exercises. Almost every chapter has a large section on the history of the theory of equations that highlights notable mathematicians and their methods. A strength of the book is the author's ability to give a good feel for mathematical developments without burdening the reader with detail. For example, the flavour of such results as the intermediate value theorem, Euler's formula for $e^{i\theta}$, and most proofs of the fundamental theorem of algebra is given without going beyond the elementary scope of the book. Readers who wish to learn more can consult a seventy-item bibliography.

After an introductory chapter that introduces polynomials and their graphs, Irving begins on familiar territory with a complete discussion of quadratic equations. One pleasant feature of this chapter is that he does not merely state (as is often done in school courses) that the graph of a quadratic is a parabola, a curve that students generally have no experience with, but verifies that the graph has the focus-directrix property; he provides a geometric interpretation of completion of the square. The same thorough treatment occurs in subsequent chapters in which he treats the cubic and quartic equations. Not only is there a comprehensive treatment of the several methods of solution, but he goes into significant detail on the discriminant and the character of the roots. A separate chapter is devoted to the basics of the complex field and its geometric realization.

Where do we go from here? Not to Galois theory, as one might expect, but to the issue of polynomials of any degree having a root. The final chapter is masterly. After a historical survey of progress on the quintic equation, the author turns to the fundamental theorem of algebra. He briefly describes a number of approaches to the theorem, but sets his sights on a less common proof that uses induction on the exponent of the highest power of 2 that divides the degree of the polynomial. For this, he develops the basic theory of fields and field extensions, rings of polynomials and the relationship between coefficients and symmetric functions of the roots. The idea is to construct the splitting field, apply the induction hypothesis to the polynomial whose roots are $r + s + mrs$, where r, s run through pairs of roots of the original polynomial, and show that $r + s$ and rs are actually in the base complex field for some pair.

Irving's book can be used by a secondary teacher looking for material that can be handed to a curious student who wants to go beyond the current slender syllabus.

Cuoco and Rotman, on the other hand, have produced an updated version of a standard first modern algebra course at the tertiary level. The definitions, theorems and proofs are laid out more formally, but there are many enhancements to the treatment. They are sensitive to historical developments and the book is punctuated by historical notes. They advise students on how to assimilate material in sections entitled "How to think about it". Side notes in the margin elucidate notation and vocabulary, comment on the treatment and provide context. Unlike the Irving book in which exercises are woven into the treatment, here exercises occur at the end of sections. On occasion, they preview later material or invite readers to follow up some line of enquiry on their own.

In contrast to Irving with his greater emphasis on algorithms, Cuoco and Rotman are more concerned with the structures of algebra. However, they start close to the ground with an opening chapter on early number theory that takes us through some Babylonian problems, Pythagorean triples and Diophantus' method to find them, the Euclidean algorithm and the fundamental theorem of arithmetic. A chapter on induction provides an opportunity to cover diverse applications of the method. A brief treatment of the solution of cubics and quartics leads to an extended section on the complex plane, roots of unity, and Gaussian and Eisenstein integers.

A detailed treatment of modular arithmetic is enriched by the inclusion of the basics of ring theory, public key codes, Conway's method for determining day of the week and patterns in decimal expansions. Finally, we get to the structures of modern algebra: rings, domains, fields, ideals and homomorphisms. The ring of polynomials with its analogies to $\mathbf{Z}$ along with extension fields and a sidelight on straight-edge-and-compasses constructions, is treated. After a chapter on algebraic integers and primes, we get to the main business of the book, how to approach Fermat's Last Theorem. After the cubic case is settled, we study the attempt of Kummer using cyclotomic integers and the later introduction of ideals to get around the difficulties when the cyclotomic ring does not admit unique factorization. An epilog gives a brief overview of the work of Abel and Galois on unsolvability by radicals, group theory, as well as elliptic functions and curves and their relevance to Wiles' proof of Fermat's Last Theorem.

Both of these books require substantial commitment by their readers, but the treatment is comprehensive enough that anyone with a solid secondary background will be richly rewarded by increased insight into the history and techniques of mathematics.