PASSACAGLIA ON AN ODD THEME

A mathematical vignette

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§**1.** Consider the numerial sequence: $1, 2, 3, 4, \cdots$. Strike out every second number to obtain: $1, 3, 5, 7, \cdots$. Now make a running total of its terms: $1, 4, 9, 16, \cdots$. We get the perfect squares. This can be demonstrated by a diagram, a square array of dots that can be counted along successive gnomons. We can conclude by asserting that the sum of the first n odd integers is n^2.

Now write the numbers in groups, with each group containing one more number than its predecedssor: $(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), (11, 12, 13, 14, 15), (16, 17, 18, 19, 20, 21), \cdots$. Strike out every seoond groups: $(1), (4, 5, 6), (11, 12, 13, 14, 15), (22, 23, 24, 25, 26, 27, 28, 29), \cdots$. Now write the sum of each group in turn. One can of course add the numbers one by one. However, there is an alternative approach. Consider for example, the group $(11, 12, 13, 14, 15)$. If we take 1 from 14 and add it to 12, 2 from 15 and add it to 11, we do not change the sum, which will not be equation to the sum of $(13, 13, 13, 13, 13)$, namely $5 \times 13 = 65$. So we now obtain the sequence: $1, 3 \times 5 = 15, 5 \times 13 = 65, 7 \times 25 = 175, \cdots$. Now take a running total: $1, 1 + 15 = 16, 1 + 15 + 65 = 81, 1 + 15 + 65 + 175 = 256, \cdots$. These numbers turn out to be the fourth powers of the natural numbers.

Why does this happen? A formal algebraic argument requires some effort and may in the end by less than illuminating. However, we can look at matters in this way. By way of example, look at the sum of the first four groups:

$$1 + (4 + 5 + 6) + (11 + 12 + 13 + 14 + 15) + (22 + 23 + 24 + 25 + 26 + 27 + 28).$$

By using the same taking away from one and adding to another strategy employed earlier, we can get the equal sum:

$$1 + (3 + 5 + 7) + (9 + 11 + 13 + 15 + 17) + (19 + 21 + 23 + 25 + 27 + 29 + 31).$$

This is the sum of the first $1 + 3 + 5 + 7 = 4^2$ odd numbers, which is equal to

$$(1 + 3 + 5 + 7)^2 = (4^2)^2 = 4^4.$$

We can see that the same pattern works for the other running totals. This is an argument that is accessible to younger students than an algebraic proof would be; furhermore, it gives insight into the underlyhing mechanism that makes it work.

For further exploration, following the same procedure where we increase each row by other fixed amounts:

$$(1), (2, 3, 4), (5, 6, 7, 8, 9), (10, 11, 12, 13, 14, 15, 16), (17, 18, \ldots, 24, 25), \ldots;$$

$$(1), (2, 3, 4, 5), (6, 7, 8, 9, 10, 11, 12), \cdots.$$

In the first of these generalizations, then sums of the odd rows are 1, 35, 189, $\ldots$, with running totals 1,

$$\begin{aligned} 1 + 35 &= 1 + (5 + 6 + 7 + 8 + 9) = 1 + ((3 + 5) + (7 + 9 + 11)) \\ &= (1 + 2 + 3)^2 = 36. \end{aligned}$$

$$\begin{aligned} 1 + 35 + 189 &= 1 + (5 + 6 + 7 + 8 + 9) + (17 + 18 + \cdots + 24 + 25) \\ &= 1 + ((3 + 5) + (7 + 9 + 11)) + ((13 + 15 + 17 + 19) + (21 + 23 + 25 + 27 + 29)) \\ &= (1 + 2 + 3 + 4 + 5)^2 = 225. \end{aligned}$$

and so on. There is also a similar pattern for the sums of the even rows: 9, 91, and so on, where the running totals equal 1, $100 = (1 + 2 + 3 + 4)^2$, and so on.

§**2.** This can be generalized in another direction. Start with the same integer sequence:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \cdots.$$

This time strike out every third term:

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, \cdots.$$

Now make a running total:

$$1, 3, 7, 12, 19, 27, 37, 48, 61, 75, 91, 108, \cdots.$$

Strike out every second term:

$$1, 7, 19, 37, 61, 91, \cdots.$$

Now make a running total:

$$1, 8, 27, 64, 125, 216, \cdots.$$

The cubes appear.

At this point, one wonders whether we can make fourth and high powers appear. Indeed we can. To get the nth powers, begin by striking out every nth term; take a running total; strike out every $(n-1)$th term; take a running total, and continue on in this way. For fourth powers, we have, in turn

$$1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, \cdots$$

$$1, 3, 6, 11, 17, 24, 33, 43, 54, 67, 81, 96, 113, 131, 150, \cdots$$

$$1, 3, 11, 17, 33, 43, 67, 81, 113, 131, \cdots$$

$$1, 4, 15, 32, 65, 108, 175, 256, 369, 400, \cdots$$

$$1, 15, 65, 175, 369, \cdots$$

$$1, 16, 81, 256, 625, \cdots$$