INTRODUCTION

When I have had the opportunity to interact with school students, I usually go in with some problem or investigation that will engage them and illuminate some part of the syllabus. I look for situations that almost every student can make some progress in, but which have aspects that go deeper and will keep the quicker students on board.

Both mastery of procedures and logical reasoning are key parts of the curriculum, but these must be pursued in a context in which they make sense and arise naturally as a part of whatever investigation the students are engaged with. An important characteristic of student learning is being sensitive to patterns and willing to change perspective, because success in mathematics is often dependent on developing attentiveness to details and looking at things in a productive way.

School education can often lead to students going through the motions blindly. Here are a few examples of how students can get in a rut. Once I was in a Grade 5 (!) class in which a girl had to find the change when making a purchase. Without batting an eyelash, she offered an amount that was greater than the bill tendered (evidently adding rather than subtracting). It was only when I mentioned that she was getting more money back than she was giving, that, after a brief pause, she realized the situation.

An algebra student, asked to solve a quadratic equation given in the form $(x - a)(x - b) = 0$, multiplied the factors and then applied the quadratic formula. Another student, having solved an equation, asked me whether his answer was correct. I asked whether he had plugged the solution into the equation to check, and he seemed to be unaware of the point of solving an equation.[1]

Even when doing an automatic procedure, a students need an elf sitting on their shoulders, to keep them aware of what is going on, why it is appropriate, and what might be going wrong.

Here is a situation that happened to me recently that illustrates mathematical awareness (or lack thereof) in a transaction.

I made a purchase that cost \$12.15, and paid in cash, putting a \$20 bill on the counter along with \$2.15 in change. The clerk was completely confused by this.

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[1]At the college level, there was the time when I asked, unwisely as it turned out, on a test for the students to find the Fourier expansion of $\cos^2 x$ on the interval $[0, 2\pi]$, expecting them to write down $\frac{1}{2}(1 + \cos 2x)$ and recognize this as the answer. (The Fourier expansion is a representation of the function as a sum a series of certain trigonometric functions whose coefficients are given by a particular integral formula.) Some students got bogged down by the intgration required for the coefficients, while others wrote down the correct answer and then used it to perform the necessary integrations. An aware student would either have recognized that $\cos^2 x$ satisfies conditions that ensure that it had a unique trigonometric expansion which was its Fourier series, and that $\frac{1}{2}(1 + \cos 2x)$ was a trigonometric expansion, or mentally noted that because of the orthogonality conditions, the Fourier expansion of $\frac{1}{2}(1 + \cos 2x)$ was itself.

When people regularly used cash rather than cards in stores, it would be understood that the customer wanted to reduce the amount of loose change. Formerly, the cashier would have thought, "That is \$10 out of \$20; here is your ten dollar change."

Conceptually, the \$2.15 can be thought of as a down payment on the \$12.15 cost, leaving the balance of \$10 to be covered by the bill. A modern cashier, blind to this view, might either have just entered the cost and amount tendered into the cash register which would then have calculated the change (which is what happened here), or else left the \$2.15 on the counter and given the customer \$7.85 (leaving the customer with a pocketful of change). The change in each case is correctly calculated, but what a difference in the environment!

There is an interesting book by Suzanne Simard, *Finding the Mother Tree*, published in 2022. Dr. Simard is a forest ecologist and has studied trees as a community, in which the plants, even of different species, can communicate with and support each other to create a healthy forest. This might serve as model for a school curriculum. Currently, we are likely to pay attention to the individual topics (trees) at the expense of the health of the ecosystem (forest) as a whole, a system in which various topics feed into each other and different areas have structural and analogical connections that can firm up the learning and be exploited.

The problems and demonstrations that follow are ones that to varying extent I have tried with students in university teacher preparation classes and in schools. It is hard to predict what will happen; this depends on how the students respond and the ability of the teacher to deal with the twists and turns. However, I have generally had a positive experience. Students vary in their background, intuitions, their interest, their ability to think divergently, their technical skill, the perspective they bring to the problem and their ability to articulate their reasoning.

I invite teachers in schools and at educational institutions to try some of the problems with their classes; I would like to receive reports on what transpired, sent to barbeau@math.utoronto.ca.

More material can be found on my website https://www.math.utoronto.ca/barbeau/home.

SOME PROBLEMS AND INVESTIGATIONS

Problem 1. Using each of the eight digits $1, 2, 3, 4, 5, 6, 7, 8$ exactly once, create two numbers whose difference is as small as possible.

Discussion. This can begin with simply asking for the pupils to make up two numbers and find the difference, and then once things get going, asking to minimize the difference. This is a nice task to get a sense of where the various students are at. How strategic are they in getting to the answer? How long does it take for the pupils to, implicitly or explicitly, realize that the minimizing pair should each have four digits? Can anyone explain why their answer is correct?

It would be interesting to know if students take for granted that the two numbers should each have four digits, or whether they try other possibilities first. When is it worth raising the issue of the number of digits of the two numbers?

Other sets of digits can be given, including ones where some digits are repeated. One might consider $(1, 2, 3, 4, 6, 7, 8, 9)$ for example.

The same question can be asked to minimize or maximize the sum or product of the two numbers, or to get the quotient as close to 1 as possible.

Problem 2. (a) Using each of the ten digits $1, 2, 3, 4, 5, 6, 7, 8, 9, 0$ exactly once, make up three numbers a, b, c for which $a + b = c$.

(b) Using each of the nine digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once, make up three numbers A, B, C for which $A + B = C$.

Discussion. Since this is a rather tedious search with many possibilities, it is probably best given as a sleeper problem, where students write the answers on a chart as they find them. One can start by noting that c must have exactly 4 digits (why?). The wrinkle with this is to recognize that there are two possible configurations, one where a and b each have three digits and the other where a has four and b has two digits. In the latter case, this should tell us something about the first two digits of c.

One way to start is to note that, for (a), the smallest sum c is at least 1023 and look and see whether this number or the next few numbers is possible.

Problem 3. Find three three-digits numbers, where each of the nine non-zero digits is used exactly once, for which they are in the ratio $1 : 3 : 5$.

Discussion. The last digit of the largest number is clearly 5; the fact that larger two numbers cannot begin with 5 puts a restriction on smaller two numbers. This could be called the *hymn board problem*, but students who are not churchgoers will not be familiar with the setting.

Problem 4. Find all the integers exceeding 1 that are the sums of the squares of their digits; the cubes of their digits.

Discussion. You can put an upper bound on the candidates by noting that the digits cannot exceed 9. If the number has a digits it lies between 10^{a-1} and 10^a while the sum of the kth powers of its digits cannot exceed $a \times 9^k$. In the case of the cubes, there are four possibilities.

Extension. Let $f(n)$ equal the sum of the kth powers of the digits of n. We can define the *orbit* of f to be a sequence produced by applying f over and over again: n, $f(n)$, $f(f(n)$, ... For example, when $k = 3$, we get an orbit

$$32 \to 35 \to 152 \to 134 \to 92 \to \ldots.$$

Problem 5. Find a ten-digit number $ABCDEFGHIJ$ whose ten digits are all different for which A is a multiple of 1, AB is a multiple of 2, ABC is a multiple of 3, ..., $ABCDEFGHI$ is a multiple of 9 and the number itself is a multiple of 10.

Discussion. The attraction of this problem is that there is a variety of types of information that you can pick up about the number that allows you to narrow down the prospects. As a bonus, it turns out that there is exactly one possibility. It is immediately clear what J and E must be. Every second digit must be even, so that the remaining digits are all odd. We do not have to worry about A and I since the numbers ending in these digits are automatically divisible by 1 and 9, respectively.

In the course of the search, it can be developed that any number is divisible by 4 if and only if the number formed by the last two digits is divisible by 4 and is divisible by 8 if and only if the number formed by the last three digits is divisible by 8. Can any student explain why this is so? This says that D and H are 2 and 6 in some order.

Finally, $A+B+C$, $D+E+F$ are both divisible by 3. Putting all this together yields a small number of cases to be checked.

Problem 6. Find all positive integers $ABC\dots$ with ten or fewer digits for which A is the number of zeros, B is the number of ones, C is the number of twos, and so on, in its numeration.

Discussion. This can start off with some trial and error, in which is can be learned that A cannot be 0. The possibility that $A = 1$ allows students to test the waters and get a feel for the situation; an answer can be arrived at quite readily.

Another interesting approach is by a method of "successive approximation". Start with a guess. Refine this by a second guess in which A, B, C, ... count to number of zeros, one, twos, and so on in the first guess. Continue on. This is hit or miss; either you will hit an answer, or you will hit a cycle. This is an like a standard mathematical strategy used in approximately solving algebraic or differential equations. For example

$$111223 \to 032100 \to 311100 \leftrightarrow 230100.$$

Extension. We can study the finite sequences $(w_0, w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ for which w_i is the number of times that i occurs in the sequence. What can be said about the sum of the terms in the sequence? Give expressions for this sum.

Problem 7. Find two distinct pairs (a, b) and (c, d) of positive integers for which the sum of each pair is equal to the product of the other pair.

Discussion. Since many pupils will know that $2 + 2 = 2 \times 2$, it might be worth asking what happens when the pairs are the same. Then $a + b = ab$. Suppose $a \geq b$. What does this tell us about b? What happens when we allow a and b to be fractions?

If students are unable to get started, it may be worth just coming up with an arbitrary example for one of the pairs. For example, is it possible for one of the pairs to be $(3, 5)$ and for the second pair to have product 8 and sum 15? This may lead to the insight that for one of the pairs, either the product and sum are the same, or the product is less than the sum. What does this tell us?

What happens in the special cases that b is equal to 1 or 2, for example?

One way to approach this is to subtract the two equation to get

$$a + b - ab = cd - c - d,$$

which can be rewritten as

$$(a - 1)(b - 1) + (c - 1)(d - 1) = 2.$$

Extension. We can allow the integers in each pair to be not all positive. How many of them can be negative? What happens if one of them is zero? Suppose the product of each pair is twice the sum of the other pair?

Problem 8. Find unequal positive real numbers a and b for which $ab = a^b$.

Discussion. One way of starting is to see if any examples can be found when a or b is a positive integer. An interesting case is $(a, b) = (\sqrt{3}, 3)$; if this is found, this along with $(a, b) = (2, 2)$ may suggest a pattern. Algebraically, we can rewrite the condition as $a = b^{-(b-1)}$; setting in different values for b and checking the results given en exercise in manipulating exponents.

Problem 9. Find some unequal positive real numbers a and b that satisfy $a^b = b^a$.

Discussion. This is a nice situation to check on the facility of student with exponents. The key is to put the exponents over a common base by setting $b = a^m$ for some m. This leads to $a^{m-1} = m$ from which $(a, b) = (m^{1/(m-1)}, m^{m/(m-1)})$. Students can check this with particular values of m such as 2, 3 and 1/2.

For secondary students, more insight to the setting can be given by studying the function $(\log x)/x$ and seeing when it takes the same value at two different values of x.

Problem 10. (a) Verify, by factoring the numbers in each equation, that $3516 \times 8274 = 4728 \times 6153$ and $992 \times 483 \times 156 = 651 \times 384 \times 299$.

(b) Find all the numbers AB and CD (with digits A, B, C, D) you can for which $AB \times CD = DC \times BA$.

Discussion. (a) is not as onerous as it might seem, since some factors are easy to find (such as powers of 2 and 3) and any factor on the left side is also a factor on the right. This helps with (b). For example, if you try $AB = 13$, then $BA = 31$ and so CD must be a multiple of 31.

Extension. As a sleeper problem, one can ask for examples where at least one of the numbers has more than two digits.

Problem 11. Which positive integers can be expressed as the difference of two squares (one of them could be 0)?

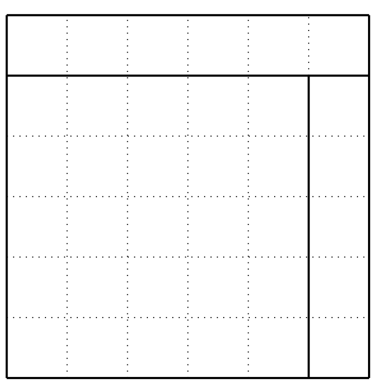
Discussion. You could begin with telling the children to find integers that are the difference of two squares. This allows the pupils to generate their own examples while giving a modest amount of arithmetic practice. It is not hard to find a representation of a given number as a difference of squares (if such exists), and when enough information is available, there is a pattern to be seen, to wit, that a representation is possible as long as the number is not twice an odd number.

One issue that might arise is how large the squares you need to test with when looking to represent a given number as a difference of squares. For example, if you want to represent 75 as a difference of squares, is there an upper bound to the squares that need to be checked?

There are a number of possible paths: (1) representing odd numbers as a difference of squares; (2) representing multiples of 4 as a difference of squares; (3) arguing that a representation is not possible for even numbers not a multiple of 4.

The question for the teacher is whether and when the children cotton on to the fact that you get the odd numbers by taking the difference of consecutive squares and the multiples of 4 as the difference of consecutive squares of the same parity.

Some children may be interested in the fact that the difference between consecutive squares is the sum of their roots and this is the opportunity to present some kind of justification. One possible approach is to use an illustrative diagram for the spcific case $6^2 - 5^2 = 6 + 5$.



Another approach is to build on the idea that to get from 5^2 to 6^2, you need to add 5 to get from 5×5 to $6 \times 5 = 5 \times 6$ and then 6 more to get from 5×6 to 6×6.

If the students have had introductory algebra, then they will be familiar with the difference of squares factorization $x^2 - y^2 = (x - y)(x + y)$. The significant property to focus on is that the difference of squares is the product to two integers *of the same parity.* So the question of which numbers are representable as the difference of squares reduces to which numbers can be represented as the product of two numbers of the same parity. (Probably the most useful algebraic identity that students learn is the formula for the difference of squares, which can be applied in a remarkable number and variety of situations.)

Another approach to the representation question is to note that each square leaves remainder of 1 or 0 upon division by 4, and so it is not possible for the difference to leave a remainder of 2.

Extensions. We can ask how many representations a number has as a difference of squares, and whether there are any numbers for which there is only one possibility. As we have seen, it is a question of the number of ways an integer can be factored as a product of two numbers of the same parity.

One can also ask which positive integers can be expressed as the sum of two integer squares. This is a much more difficult question involving a proof that is beyond the scope of school students. This provides an opportunity to talk a little history. However, it is still worth posing as there are a number of things that students might observe: the primes for which such a representation is possible, situations in which there is more than one representation. Algebra students can be shown the formula that allows a representation for a product when the representation for each factor is known. One demonstration that is accessible at the school level is the algorithm of Zagier to determine representations of primes (congruent to 1 modulo 4) as the sum of two squares.

Problem 12: Positive integers equal to the sum of consecutives. The question is simple: which positive integers can be expressed as the sum of at least two consecutive integers?

Discussion. The steps in this problem might include starting with odd integers as the sum of two consecutives, recognition that the sum of an odd number of consecutive integers is the middle one times the number of integers and that an even number of consecutive integers can be summed by pairing them working from opposite ends. In this way, students might arrive at the key idea that the representation of an integer as a sum of consecutives is related to its divisibility.

If the students make a table showing representations, then it becomes apparent that the only nonrepresentable integers seem to be powers of 2.

Problem 13: Equal representation of a number by the sums of two adjacent sets of consecutive integers. We start with $1 + 2 = 3$ and $4 + 5 + 6 = 7 + 8$. In each case, a set of consecutive integers can be split into two subsets of consecutive integers with the same sum. Find other examples of this.

Discussion. We can begin with the special case that there is one more integer on the left side of the equation than on the right. There are a number of approaches that might occur. One is simply to put more information on the table, and pattern recognition might lead to the additional examples

$$9+10+11+12=13+14+15;$$

$$16+17+18+19+20=21+22+23+24.$$

There are at least two ways the justification of this can go. One is to note that each equation starts with a square, and verify one of the equations in a way that makes the truth of them all evident: Possibilities are

$$(13-12)+(14-11)-(15-10)=1+3+5=3^2=9$$

or

$$(15-12)+(14-11)+(13-10)=3+3+3=3\times 3.$$

If the students have studied algebra, then it is a matter of constructing an equation involving a variable. Then it is a question of selecting them appropriately. Suppose that there are $m+1$ terms on the left side and m terms on the right. While one possibility is to let the smallest integer in the equation be n, a more efficient choice is to let n be the middle term. Then the general equation is

$$(n-m)+(n-m+1)+\cdots+(n-1)+n=(n+1)+(n+2)+\cdots+(n+m).$$

This leads to

$$n=2[1+2+\cdots+m]=m(m+1).$$

The investigation may then be extended to finding cases where the sum on the left has some fixed number of terms more than the sum on the right or twice the number of terms on the left as on the right, such as

$$2+3+4+5+6+7=8+9+10.$$

Extension. Is it possible to have a set of consecutive integers that can be split into three sets of consecutive integers all with the same sum?

Problem 14. Construct as many pythagorean triples (a,b,c) (for which $a^2+b^2=c^2$) as you can.

Discussion. For some reason, many students are exposed to pythagorean triples in middle school; some seem to be intrigued by them and hunt for others. The point here is to see what patterns they can find and how families of triples can be expressed by an algebraic equation.

Of particular interest are triples where two of the three numbers differ by 1, either since $c=b+1$ or $b=a+1$. Triples such as $(3,4,5)$, $(5,12,13)$, $(7,24,25)$ have a ready description: take an odd number, square it and write the square as the sum of two consecutive integers.

More interesting are the cases where $b=a+1$. Finding examples beyond $(3,4,5)$ is more difficult, and a pocket calculator or a computer would be useful. However, $(20,21,29)$ is reasonably accessible. Once $(3,4,5)$ and $(20,21,29)$ are in hand, one can look for patterns such as:

$$5=1^2+2^2;\quad 5-4=(2-1)^2=1^2;\quad 5+4=(2+1)^2=3^2; 1\times 3=3;$$

$$29 = 2^2 + 5^2; \quad 29 - 20 = (5-2)^2 = 3^3; \quad 29 + 20 = (5+2)^2 = 7^2; 3 \times 7 = 21.$$

The equation $a^2 + (a+1)^2 = c^2$ can be rewritten as $(2a+1)^2 - 2c^2 = -1$, which is an example of a Pell's equation: $x^2 - 2y^2 = -1$. It is not too onerous for students to find some solutions of this equation: $(x, y) = (1, 1)$, $(7, 5)$, $(41, 29)$ and generate the corresponding pythagorean triples.

Another direction in which this can be taken is to relate a pythagorean triple (a, b, c) to the point $(\frac{a}{c}, \frac{b}{c})$ in the cartesian plane. Such a point can be written in the form $(\cos\theta, \sin\theta)$. Then the trigonometric identities

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi)$$

and

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi)$$

can be used to generate additional pythagorean triples. (These formula induce a "multiplication" on the pythagorean triples:

$$(a, b, c) * (u, v, w) = (au - bv, av + bu, cw).$$

Extension. The diophantine equation $x^2 - 2y^2 = \pm 1$ leads us into a rich ecosystem of relationships. One way to approach it is to present students with the following table, and ask them to look for patterns and decide how the table can be extended:

1	0	0
1	1	1
3	2	6
7	5	35
17	12	204
41	29	1189
99	70	6900

In particular, look at how it relates to the pythagorean triples $(3, 4, 5)$, $(20, 21, 29)$, $(119, 120, 169)$, $(696, 697, 985)$.

The introduction of trigonometric functions provides the occasion to give students a more robust introduction to complex numbers and show the students an application of de Moivre's Theorem:

$$(cos\theta + i\sin\theta)(cos\phi + i\sin\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi).$$

Problem 15. In the equation $3^2 + 4^2 = 5^2$, we have squares of consecutive integers and one more term on the left than on the right. What other equations can you find involving squares of consecutive integers where the number of terms on the left side is one more than the number of terms of the right?

Discussion. There is one obvious answer that is very likely to be missed by the students:

$$(-k)^2 + (-k+1)^2 + \cdots + (-1)^2 + 0^2 = 1^2 + 2^2 + \cdots + (k-1)^2 + k^2.$$

Are there other examples? A little experimenting might turn up the example $10^2 + 11^2 + 12^2 = 13^2 + 14^2$, which provides the opportunity to look for patterns and conjecture more examples. I have found that students focus on the terms closest to the equals sign and on the left term and can turn up a third example $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

There are two insightful ways to see why it works, illustrated by the last equation:

$$(27^2 - 22^2) + (26^2 - 23^2) + (25^2 - 24^2) = 49 \times (5 + 3 + 1) == 49 \times (3 \times 3) = 21^2;$$

$$(25^2 - 22^2) + (26^2 - 23^3) + (27^2 - 24^2) = 3 \times (47 + 49 + 51) = 3 \times 3 \times 49 = 21^2.$$

An alternative approach is to use some algebra. One can take the smallest square to be n^2 with $k+1$ terms on one side and k terms on the other. However, a more efficient choice is to let n be the middle number, so we are looking at the equation

$$(n-k)^2 + (n-k+1)^2 + \cdots + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2 + \cdots + (n+k)^2.$$

When the dust settles we see that we have a monic quadratic equation in n for which the constant term is 0 (we would expect this since we know that 0 is one of its roots). We get $n^2 = 4n(1 + 2 + \cdots + k) = 2nk(k+1)$ and $n = 2k(k+1)$.

Extension. Look for equations in which one sum of consecutive squares is equal to the other, but the number of terms and the closeness of the two sets of squares varies.

Problem 16. Make up a table so then in each row, you list an integer n, the number 2^n and the number 5^n. As you read down the 2^n and 5^n columns, the numbers get bigger, but sometimes two consecutive powers have the same number of digits. What do you notice about when this happens and how do you account for it?

Discussion. Here is the table:

n	2^n	5^n
1	2	5
2	4	25
3	8	125
4	16	625
5	32	3125
6	64	15625
7	128	78125
8	256	390625
9	512	1953125
10	1024	9765625

Every time we increase the exponent by 1, *exactly one* of the two powers gets longer by one digit. Both powers cannot increase their length together, nor can they both keep the same length. The key to explaining the phenomenon is to note that a positive integer m has exactly k digits when $10^{k-1} \leq m < 10^k$.

Extension. There is a related phenomenon. Express the powers of 10 to base 2 and to base 5. For example, since

$$\begin{aligned} 1000 &= 512 + 256 + 128 + 64 + 32 + 8 \\ &= 1 \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 \\ &\quad + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 0 \times 1, \end{aligned}$$

we can write 1000 in base 2 numerations as $(1111101000)_2$, where it has 10 digits. Since

$$1000 = 625 + 375 = 1 \times 5^4 + 3 \times 5^3 + 0 \times 5^2 + 0 \times 5 + 0 \times 1,$$

we can write 1000 in base 5 numeration as $(13000)_5$ with five digits.

If we write out all the powers of 10 in these two bases, for each whole number m greater than 1, there is a power of 10 that has a representation with m digits in *exactly one* of the two bases 2 and 5. Successive powers of 10 in base 5 have 2, 3, 5, 6, 8, 9, 11, ... digits, while successive powers of 10 in base 2 have 4, 7, 10, ... digits ($10 = (1010)_2$, $10^2 = (1100100)_2$).

Problem 17. There is a well-known problem that asks the location of a man who walks one mile south, then one mile west, and finally one mile north and ends up where he started.

Discussion. The traditional answer is, the north pole. However, someone later noticed that this is not the only location. He could have started at a suitable distance from the south pole, walked a mile south to a certain point, and then returned to this point after a westward walk of a mile that took him through 360 degrees of latitude, and then finished by retracing his steps north to his starting point. In fact, he could have started closer to the south pole, with the westward walk taking him through any positive integer multiple of 360 degrees.

Problem 18. Consider some other person whose date of birth you know; it could be a friend or a family member. The age of a person is the whole number of years that they have lived; it is a nonnegative integer. For example, the age of a baby not yet one year old is 0. When you celebrate a birthday, your age increases by 1.

At some time, the age of the older of two people will be twice the age of the younger. What is the total amount of time that this occurs for you and the other person?

Discussion. The cleanness of the problem is slightly confounded by that fact that not all years have the same number of days; however, in practice this does not seem to be an issue. Another setting is to have two runners running at equal speeds around a circular track, one starting earlier than the other. Then the question is for how long the number or laps completed by the earlier runner is double that of the later.

The remarkable answer is that in all cases, the total amount of time will be one year, broken into two time periods when the birthdays of the two individuals are not the same. The key realization is that the difference in the ages does not change from year to year and usually takes two values (that differ by 1) according to the position of the two birthdays. If $b \geq a$ are the two ages, then the solution is indicated by the equivalence between $b = 2a$ and $b - a = a$.

Exension. We can look at situations where the age of the older person is some other integer multiple of the other. For convenience, it can be assumed that their birthdays are a half year apart. How many integer multiples are possible?

Problem 19. In the Julian calendar, every fourth year had 366 days rather than 365, in an attempt to align the calendar year with the solar year. However, as time went by, it was discovered that the solar and calendar years went out of kilter because there were too many leap years. In 1582, Pope Gregory XIII, reformed the calendar by decreeing that leap years were those divisible by 4 except for years divisible by 100 but not by 400. Thus, 1600 and 2000 are leap years, but 1700, 1800, 1900 are not.

Some people are superstitious about Fridays that fall on the thirteenth of the month. One might suppose that the that it is equally likely for the thirteenth to be any day of the week. Explain why this is not so.

Discussion. The key is to recognize that the pattern of dates recurs every 400 years (or 4800 months) (January 1, 1600 occurs on the same day of the week as January 1, 2000). It would be interesting to see how the children handle the argument.

Extension. This might be an occasion for some research on the net, that might involve the difference between the lengths of the solar and sidereal years and what further adjustments to the calendar might be necessary.

Problem 20. Nina lives in town A and Olga lives in town B. One day, at exactly the same time, each sets out to walk to the town of the other, taking the same route and walking at her own constant speed. The pass each other at noon. Nina reaches B at 4 pm, while Olga, a slower walker, does not arrive at A until 9 pm. When did the ladies set out?

Discussion. This question is quite difficult, and seems even harder for pupils who have studied algebra. These pupils try to set up an equation and get into trouble. In particular, what should the variable represent?

One way to break the impasse is to ask the class to guess the starting time and check whether it works. This may gently nudge them to the core of the problem: the ratio of the times taken for one lady to traverse the two legs of the journey (A to the passing point, the passing point to B) is proportional to ratio of the times for the other lady.

Problem 21. A store is having a sale in which the list price is discounted by 20%. A sales tax of 13% is levied on the amount paid for any purchase. Which of these procedures is correct?

(1) The sales tax is first applied to the nonsale price and then the resulting total is reduced by 20%;

(2) The sales tax is applied to the discounted price.

Discussion. This seems to be a situation in which some students may by some sort of intuition accept that it makes no difference, but for others, it might not seem obvious. The two procedures lead to the same result, as one can check on a calculator using some amount, say $100, as the nonsale price. More formally, if P is the non-sale price, then (1) leads to $P(1.13)(0.8)$ and (2) to $P(0.8)(1.13)$, which are equal by the laws of arithmetic. However, this formal approach may not lead to an appreciation of the proportionality that is involved.

A possibly more insightful way to think about it is to look at what would happen if the article is sold at the nonsale price. We can split this price into two parts: the sale price and the discounted amount. The total sale tax is the sum of the tax applied to these parts separately. Since the sales tax is proportional to the base amount, the sales tax on the discounted amount is 20% of that on the nonsale price. This gives us a visualization of the process that may be more persuasive.

A geometric picture may be useful. Draw a line AC with in inner point B located so that AB represents the non-sale price and BC the sales tax. Now draw a second line DF, which is AC reduced by 20% and B corresponding to E, an inner point of DF. DEF can be interpreted in two ways: (1) as AC the total original cost, with DF to total cost after the 20% reduction; (2) DE representing the sale price, and EF the sales tax tacked on at the end (noting that $AB : BC = DE : EF$).

Problem 22. Suppose two people play a rubber of table tennis, in which the winner is the first one who can win two games. Thus, the rubber can consist either of two games won by the same player, or of three games in which each player wins one of the first two games and the third is won by the winner of the rubber. In order to win a game, a player must obtain at least 21 points and gain at least two points more than the opponent.

A particular rubber lasted for three games, and for each player, their score for the middle game was the average of their scores for the first and third games. No player got the same score in all three games. In other words, the scores for each player are in **arithmetic progression**: the difference in score between the first and second games was equal to the difference between the second and third games for each player. What was the score for the losing player of the third game?

Discussion. It seems that the information given is a little sparse, and, indeed, there are several possible outcomes for the three games. However, it turns out that the answer is the same for all of them: 15.

To understand what is going on, we make a few observations. First, the same player cannot win the first and third games, because the average score for the winner would exceed the average score for the loser, so that player would have won the second game. If this case, the rubber would have concluded in two games.

Thus, one player, A, won the first game and the other, B, the last two. Since B's score in the last game was at least 21, in the second game it was more then 21. In the first game, even though B lost, he had more than 21 points, which means that A must have beat B by exactly two points. In the second game, since B's score exceeded 21, B must have won by exactly two points.

B's score minus A's score was -2 in the first game and $+2$ in the second. Since the scores of each player were in arithmetic progression, the same is true of the difference (4) between them. Thus B must have beat A by exactly six points in the third game. Since the difference is bigger than 2, this means that B got to 21 with a sufficient margin and the game ended. The final score was 21 for B and 15 for A.

What are the possibilities if we allow one of the players to have the same score in all three games?

Problem 23. (a) Let $p(x) = ax+b$ and $q(x) = cx+d$ be two linear polynomials. Under what circumstances is $p(q(x)) = q(p(x))$? Thus, the question is about when

$$a(cx + d) + b = c(ax + b) + d$$

(b) Given a function $f(x)$, we say that u is a *fixed point* for f if and only if $f(u) = u$. Under what circumstances do the two polynomials in (a) have the same fixed point?

Discussion. This is a nice example for an early secondary algebra class to tuck into; they do not get much of a chance to construct interesting theorems. While the technical side of it is quite straightforward, the students need to be quite fastidious in how they think about the situation, in particular, distinguishing between a conditional equation and an identity and in reading off the conclusions. It is not idle, as we shall see, that (a) and (b) be asked together.

If one of the polynomials is the identity polynomial $i(x) = x$, then $p(i(x)) = i(p(x)) = p(x)$ for every polynomial. We exclude this case from now on. Let E be the equation $p(q(x)) = q(p(x))$ is equivalent to $ad + b = bc + d$. This tells us that if E holds for any values of x then it holds identically for every value of x.

The condition can be rewritten as $(a-1)d = (c-1)b$. If $a = 1$, then either $c = 1$ or $d = 0$ and so we have that $p(q(x)) = q(p(x))$ whenever both polynomials have the form $x + k$.

The remaining possibility is that both a and c differ from 1, which brings us to (b). The fixed point u of $p(x) = ax + b$ satisfies $(a-1)u + b = 0$ and the fixed point v of $q(x) = cx + d$ satisfies $(c - 1)v + d = 0$. If either a or b equals 1, then there is no fixed point (naturally, since the graph is parallel to the line $y = x$). If a and b both differ from 1, then the two polynomials have a common fixed point when $(a - 1)d = (c - 1)b$, *i.e.* precisely when $p(q(x)) = q(p(x))$.

Problem 24. Sometimes, rather than ask for a solution, it is useful to have students work out the details of an insightful solution to a problem by an historical figure. A particular nice example is due to Diophantus, a mathematician who lived in Alexandria during the third century of the Christian and published a book of arithmetic problems that could be solved by algebraic techniques. One of these was to determine quadruples (x, y, z, w) of rationals, any pairwise product of which plus 1 is square (Book 4 Problem 20 in his series *Arithmetica*).

(a) Let one of the numbers be x. By examining $(x+1)^2$, $(2x+1)^2$ and $(3x+1)$, determine rational values of y, z, w so that $xy+1$, $xz+1$ and $xw+1$ are squares.

(b) Verify that two of $yz+1$, $yw+1$ and $zw+1$ are squares for any value of x.

(c) Determine a value of x for which the remaining one of the three quantities in (b) is the square of a rational.

Discussion. Begin with the equations

$$(x+1)^2 = x^2 + 2x + 1 = x(x+2) + 1;$$
$$(2x+1)^2 = 4x^2 + 4x + 1 = x(4x+4) + 1;$$
$$(3x+1)^2 = 9x^2 + 6x + 1 = x(9x+6) + 1.$$

Let the four numbers be x, $x+2$, $4x+4$, $9x+6$. Then

$$(x+2)(4x+4) + 1 = 4x^2 + 12x + 9 = (2x+3)^2;$$
$$(4x+4)(9x+6) + 1 = 36x^2 + 60x + 25 = (6x+5)^2.$$

It remains to deal with $(x+2)(9x+6) = 9x^2 + 24x + 13$.

Note that $(3x-4)^2 = 9x^2 - 24x + 16$.

If we equate these two expressions, then $48x = 3$ and $x = \frac{1}{16}$.

This gives the quadruple

$$\left(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right).$$

If we multiply this quadruple by 16, we get $(1, 33, 68, 105)$ and find that the product of any two plus 256 is a square.

We can generate infinitely many quadruples by comparing $(x+2)(9x+6)$ with $(3x+t)^2$ for any rational of t. The choice $t = 5$ leads to the quadruple $(-2, 0, -4, -12)$, or equivalently $(0, 2, 4, 12)$.

Problem 25. It was observed by many mathematicians that the product of any two integers in the triples $(1, 3, 8)$ plus 1 is a square. Naturally, one wondered whether one could append a fourth integer to make a quadruple for which the product of any pair plus 1 is square. Here is how Fermat (1601-1665) solved it.

(a) Let the quadruple be $(1,3,8,d)$ where $d+1$, $3d+1$, $8d+1$ are all squares. Set $d = t^2 + 2t$ (why?) and suppose $3d+1 = u^2$ and $8d+1 = v$. Factor $5d$ as a difference of squares, and from a factorization of $5d$ determine possible values for u and v in terms of t and then find a value of t to make $8d+1$ a square.

Discussion. When $d = t^2+2t$, then, if $3(t^2+2t)+1 = u^2$ and $8(t^2+2t)+1 = v^2$, then

$$5t(t+2) = (v+u)(v-u).$$

Let $5t = v+u$ and $t+2 = v-u$, so that $u = 2t-1$ and $v = 3t+1$.

Then

$$8(t^2+2t)+1 = (3t+1)^2$$

whence $t^2 - 10t = 0$ and so $t = 0, 10$.

This gives the quadruple $(1,3,8,120)$.

Euler was able to extend this to a quintuple, with the fifth number a rational, but not an integer:

$$\left(1, 3, 8, 120, \frac{777480}{8288641}\right).$$

In 1969, Baker and Davenport showed that if $(1,3,8,d)$ is a quadruple for which the product of any pair plus 1 is a square, then $d = 120$.

Problem 26. Suppose you put four distinct points on a page. There are six distinct pairs of them and you can find the distance between the points of each pair. Usually, the six numbers you get will all be different. However, this does not always happen.

Is it possible for all the six distances to be the same?

For what configuration of points do the six distances take exactly two distinct values?

Discussion. Children recognize that for three points, the only possibility for all pairwise distances to be equal is that they be vertices of an equilateral triangle. Suppose you want all distances equal. Wherever you place the fourth point, it has to make an equilateral triangle with each other pair. Two things might happen. Either they realize that this is not possible (in the plane) or some student might put the point into the third dimension to form a regular tetrahedron. The latter case gives the teacher the opportunity to carry the discussion in space.

For the two-distance problem, if we stay in the plane, it generally happens that someone hits upon the four vertices of a square. Think for a minute of what this signifies. This is a familiar figure for most children. Moreover, it involves a recognition that squares have four equal sides and two equal diagonals and that this characteristic is pertinent to the problem.

Another common first guess arises if the children start with the one-distance version of the problem, find that it is impossible, but settle on the $60° - 120°$ rhombus.

There are four configurations for which three of the points are vertices of an equilateral triangle, which, apart from the rhombus, are revealed in no particular order. At this point, I can only speculate what the children are thinking.

However, if the pupils start with an equilateral triangle, then implicity or explicitly, the question of the locus of a point equidistant from two given points comes up. The strategic decision for the teacher is how much to reveal and what assistance should be provided along the way.

There is a sixth configuration that may or may not emerge during a class period, and that is an isosceles trapezoid with three equal sides and the fourth equal to its two diagonals. To get this requires a fair bit of imagination of the possibilities. The final issue is that, when you have the six configurations on the table, how do you know that you have the lot?

On one occasion, I was amazed that the first configuration offered was the trapezoid. When I asked the student how he thought of it, he told me something I had not considered: take a regular pentagon and remove one of its vertices.

Extension. What are the possibilities when we ask that all six distances be integers. If the students are familiar with Pythagoras' Theorem, then a 3×4 rectangle provides an example. To make the problem more open, students can be challenged to find the maximum number of points in the plane any pair of which are an integral distance apart.

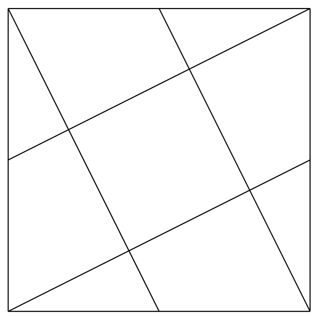
Problem 27. How many distinct points can you put in the plane in such a way that the distance between any pair of them is an integer? Suppose that all the distances should be distinct integers?

Discussion. This is probably best given as a sleeper problem that people work on over time. Perhaps there should be a "best example" that a student might enter on a bulletin board, to be superseded by an even better example. Probably the simplest example for four points is the set of vertices of a 3×4 rectangle; it is a little harder to make the points distinct.

However, some clever chap may blow the problem out of the water by putting all the points along a straight line, so we should exclude this case. The case in which all points by one are on the same straight line is more interesting. On way is to make the altitude from the point to the line one leg of a right triangle. For example 12 is involved in four pythagorean triples $(5, 12, 13)$, $(9, 12, 15)$, $(12, 16, 20)$ and $(12, 35, 37)$, and we can get an example with ten distinct points. This brings us into a side investigation of the number of ways a given number can be a leg of a pythagorean triple.

Finally, we should look at the case where no three points are collinear.

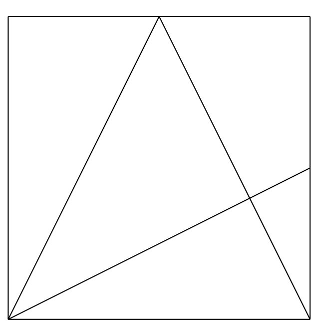
Problem 28. In the diagram below, a unit square whose area is 1 is partitioned by midlines joining each vertices to the midpoints of one of its opposite sides. What is the area of the quadrilateral in the middle bounded by the four midlines?



Discussion. This problem can be solved in a number of ways, but the most insightful way is to rotate the small right triangle at each vertex of the square so that its hypotenuse rests on the other half of the side to which it belongs.

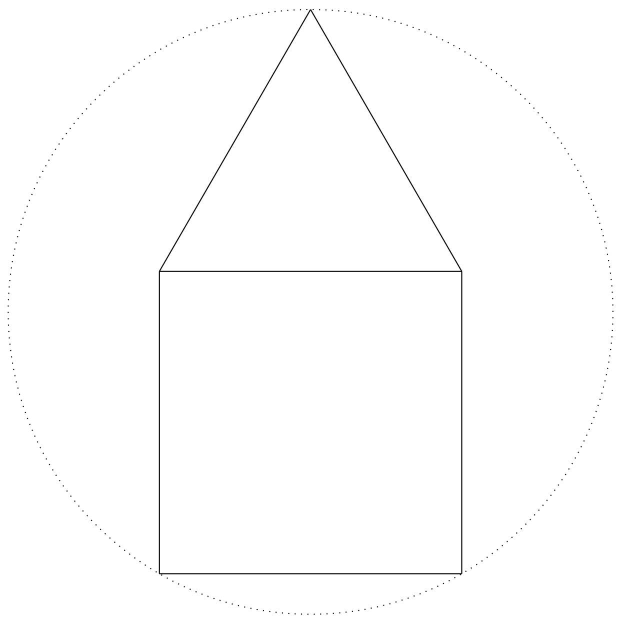
Invariably, the students will take for granted that the central figure is not just a quadrilateral but a square. Since this appears pretty obvious, one need not belabour the point. However, in one class, a student asked how we knew that it *was* a square. I put the question to the class and the students were unable to come up with an argument. Finally, I asked whether there was anyone who did not think it was a square, and no one put up their hand. What was it that made it so obvious? Finally, a student offered the word "symmetry", and this was followed by a discussion that analyzed the particular symmetry involved (a 90° rotation about the centre takes the diagram to itself).

Extensions. One can look at various ways the square can be partitioned by a selection of its midlines and ask for areas of its parts. Students may try to use analytic geometry or construct a standard Euclidean-type proof; however, they should be encouraged to think in more holistic transformational ways. One particular example involves this partition, in which each internal line joins a vertex to the midpoint of an opposite side:



Students get stuck with this one until it is realized that the right triangle along the bottom side is split into two similar triangle whose areas are proportional to the squares of their linear dimensions.

Problem 29. The diagram depicts a unit square abutting an equilateral triangle, all sides of length 1. There is exactly one circle that passes through the apex of the triangle and the two vertices at the base of the square. What is its radius?



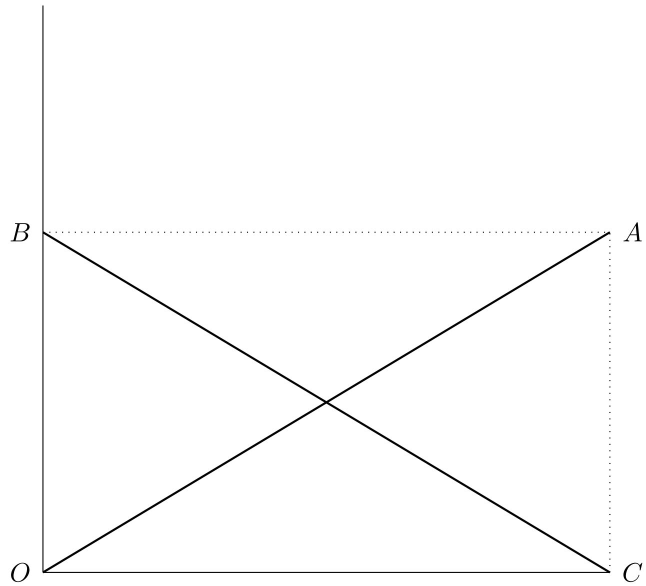
Discussion. This is a problem that students will try to tackle using either a standard school Euclidean argument or analytic geometry. But its solution lends to some insight. Imagine that the triangle atop the square slides to rest on the bottom. The apex now is unit distance from the bottom vertices and has fallen unit distance from its original position.

Problem 30. Suppose that a 2 meter ladder AO is resting flush against a vertical wall, with A the top and O the bottom resting where the wall and ground meet. The ladder then tips, with the bottom remaining stationary and the top pivoting to the ground. It is clear that the path of the top of the ladder is a circular arc with centre O and radius 2 meters. Likewise, the path traced out by a point in the middle of the ladder is a circle with radius 1 meter.

Now, suppose a different ladder BC is resting flush against the same wall, but this time, in sinks to the ground with B sliding down the wall and C moving along the ground away from the wall. What path is traced out by a point in the middle?

Discussion. Remarkably, it is the same as with the first ladder: a circle with radius 1 meter centred at the junction of the wall and ground.

One way to see this is to imagine that the two ladders fall to the ground simultaneously in such a way that their tops are at the same level all the way down:

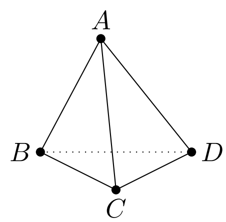


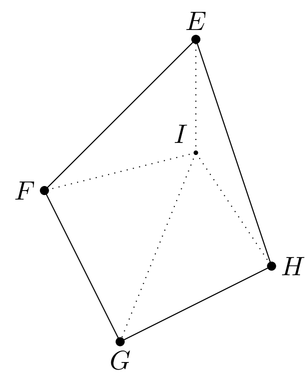
Since the lengths of the two ladders are the same, the angles they make with the ground are also the same and AC is perpendicular to OC. Thus $OCAB$ is a rectangle with diagonals OA and BC, which bisect each other. The midpoint of OA is in the same position as the midpoint of BC at each stage in the descent of the ladders.

This can be demonstrated by a fold-up ironing board. The underside of the ironing board plays the role of the wall; tucked up against it is a pair of crossing legs OA and BC with O attached to the board and B free to move along a track. The intersection of the two legs as they unfold traces out the circular arc.

The situation involves a familiar property of right triangles, like AOB. The midpoint of the hypotenuse is the centre of its circumcircle and so equidistant from A, O and B.

Problem 31. In pyramids $ABCD$ and $EFGHI$ shown below, all faces except $FGHI$ are equilateral triangles of equal size. If face ABC were placed on face EFG so that the vertices of the triangles coincide, how many exposed faces would the resulting solid have?





Discussion. This problem has a history. It was a multiple-choice question on a 1980 Educational Testing Service examination; the five options for the answer were 5, 6, 7, 8, 9. According to the examiners, the answer was 7. There are 9 faces on the two solids separately and adjoining them covered one from each solid.

However, one of the examinees objected. The answer should be 5. It turns out that when the adjunction is made, two of the faces of the tetrahedron $ABCD$ coplanar with two of the (Egyptian) pyramid $EFGHI$. One way to see this is to imagine that you have two pyramids with their square bases abutting on the table along BC. You can fit between them the tetrahedron with one edge EF coinciding with BC and the opposite edge joining the apexes of the pryamids. (The configuration with the tetrahedron and pyramid is suggestive of a puptent.)

Problem 32. In the game of noughts-and-crosses, does one player have a winning stratgy or can both players force a draw?

Discussion. This problem gets us into the recently studied area of game theory. It is familiar to many students and can be used as an example to discuss some pertinent aspects of two-person games. It is a game of strategy (no chance elements) with perfect information (each player knows the complete past history of the game at each turn, unlike card games in which each player is ignorant of the hand dealt to the other). The players take alternate turns, with the X player going first and the O player second.

There is a difference in the behaviour of students playing the game. Some seem to select the cell to play in randomly. Others play to win but not pay much attention to what the opponent is doing. Finally, there are the players who play strategically, trying to win while blocking their opponent. Since there are at most nine moves, the game must terminate, either with a win for one player or with a draw (in which no row, column of diagonal has all its cells chosen by the same player). It is not possible for both players to win.

We can form a *game tree* that lists all the ways in which the game can proceed. The opening player has essentially three options: pick the centre cell, a corner cell, or a mid-side cell. For each of the options, the second player has a similar choice; we can draw a line between each opening move to the possible second moves. A *strategy* is a path through the game tree, which gives one way in which the game progresses; the path concludes either when one player has won or when after nine moves no player has won. A player has a *winning strategy* if he can indicate a choice for his own moves such that, no matter what move the opponent makes, he can ensure a win.

Analyzing this game is reasonably taxing, involving some hypothetical reasoning, but is possible for a class to work out within a period.

One value of this game is that it illustrates a common mathematical gambit of reducing a complicated situation down to its essentials. Here, we can recast the game as one in which each player makes a single move. Imagine at the start of the game, each player writes down his complete strategy (In my first move, I will do A; if my opponent picks B, I will do C; if he picks D, I will do E and so on.) Rather than play the game, each player gives his strategy to a referee, who then constructs the game and declares the result. Thus, each player has one move: *pick a strategy.*

We then construct a *game matrix.* The first player makes a vertical list of his strategies and the second a horizontal list along two sides of a square array. The element in the ith row and jth column records the result of the game if the corresponding strategies are chosen. Any row that contains all wins for the first player indicates a winning strategy i; similarly a column that contains all wins for the second player indicates a winning strategy j for that player.

There is an interesting argument that the second player does not have a winning strategy; the game must conclude in either a draw or a first player win if both players select their best strategy. Suppose the second player has a winning strategy. This means that he can force a win, whatever the opening move of the first player. This means that in another game, the first player can play this strategy being denied any cell, and *a fortiori* with no limitations.

One might ask whether, by making a bad choice for the first move, there is a way for the second player to force a win.

There is another avenue into this topic that illustrates the idea of *isomorphism.* The mathematical structures are isomorphic if they are essentially the same; the components of one can be related to the components of the other in such a way that we can translate any characteristics of one to corresponding characteristics of the other.

In this approach, we invite student to play the following game. The first player picks one of the digits from 1 to 9 inclusive. The then play alternately, each player selecting a digit not already taken. The first player to find among his chosen digits three that add to 15 wins the game (they need not be chosen consecutively). If no player can find three such digits among his chosen ones, the game is a draw.

Students are unfamiliar with this game and so the play is uncertain. However, it is really noughts-and-crosses in another guise. To make the connection, students need to construct a magic square, a square 3×3 array containing each digit from 1 to 9 exactly one is such a way that each row, each column and each diagonal has the same sum, 15. The key observation is that the triples along these rows, columns and diagonals correspond to all the different ways you can add three digits to 15. A choice of a number by a player corresponds to placing her symbol (X or O) in that position on the magic square. Picking three numbers that add to fifteen is equivalent to picking three cells along a row, column or diagonal. This is a nice example of two structures that are *isomorphic*.

Problem 33. The Strauss-Erdös conjecture dates back to 1948. It states that, for any integer $n \geq 3$, the fraction $4/n$ can be written as the sum of three distinct reciprocals of integers. For example:

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{15} + \frac{1}{210}.$$

How far can you go in resolving the conjecture?

Discussion. Despite the fact that this is an unsolved problem, the entry point for students is surprisingly broad. Students can generate their own examples for small values of n (thus getting practice manipulating fractions).

There are a number of observations that might be made. (1) If you have a representation for a particular value of n, you can derive a representation for every multiple of n. (2) The problem can be reformulated as looking at possible common denominators for the right side and asking which of these can be written as the sum of three of its divisors. (3) The cases that n is even, is a multiple of 3 and is one less that a multiple of 4 are approachable. (4) There is a crucial identity for expressing a reciprocal as the sum of two reciprocals that is particularly useful:

$$\frac{1}{r} = \frac{1}{r+1} + \frac{1}{r(r+1)}.$$

There is a lot of pattern recognition going on here, and students may hit upon (4) empirically. For example, they may be already familiar with or easily find $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ and $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ and generalize from there. Once patterns for certain classes of integers are recognized, they can be expressed in an algebraic form.

I have found that, in a period, a class can settle the conjecture for all integers except those that are one more than a multiple of 24 (for which is is still open; it has been checked for numbers up to 10^{17}).

At this point, one can bring in the pocket calculator or program a computer to deal with larger values of n. For example, what is a representation for $n = 73$? $n = 97$? (A small side question is to find the smallest number of the form $24m + 1$ that is not a prime nor a square.)

Problem 34. Suppose that we have two positive common fractions each strictly less than $\frac{1}{2}$. It is clear that their sum is also strictly less than 1. An example is the pair $(\frac{2}{5}, \frac{3}{7})$. We observe that, while $\frac{3}{5} + \frac{3}{7} = \frac{36}{35} > 1$, $\frac{2}{5} + \frac{4}{7} = \frac{34}{35} < 1$. Is it always the case that we can increase the numerator of one of the fractions by 1 and still have two fractions whose sum is less than 1?

Discussion. If $\frac{a}{b}$ and $\frac{c}{d}$ are the two fractions, we can restrict discussion to the situation where

$$\frac{a}{b} < \frac{1}{2} \leq \frac{a+1}{b}$$

and

$$\frac{c}{d} < \frac{1}{2} \leq \frac{c+1}{d}.$$

The next reduction we can makes is that, if one of the fraction has an even denominator, then the answer is trivially yes, since we can augment the numerator to half the denominator. So we only have to consider the pairs

$$\left(\frac{a}{2a+1}, \frac{c}{2c+1}\right)$$

where a, c are positive integers with $a \leq c$. Examining numerical examples (or intuitively noting that increasing the numerator of the fraction with the smaller denominator gives a larger increase of the sum), we see that we need to check out

$$\left(\frac{a}{2a+1}, \frac{c+1}{2c+1}\right).$$

From this point on, it is algebra:

$$\begin{aligned} 1 - \left(\frac{a}{2a+1} + \frac{c+1}{2c+1}\right) &= \left(\frac{1}{2} - \frac{a}{2a+1}\right) + \left(\frac{1}{2} - \frac{c}{2c+1}\right) - \frac{1}{2c+1} \\ &= \frac{1}{4a+2} + \frac{1}{4c+2} - \frac{2}{4c+2} \geq 0. \end{aligned}$$

If $a = c$, the sum of the augmented pair is (obviously, why?) exactly 1; if $a > c$, the sum of the augmented pair is less than 1.

Problem 35. The quadratic equations $x^2 - 5x + 6 = 0$ and $x^2 - 5x - 6 = 0$ both have integer solutions. Find other positive integers a and b for which $x^2 - ax + b = 0$ and $x^2 - ax - b = 0$ both have integer solutions.

Discussion. Depending on how this problem is approached, it takes us in a number of interesting directions. For example, looking at the discriminants, we see that both $a^2 - 4b$ and $a^2 + 4b$ have to be squares, so that we can look for three squares in arithmetic progression, such as $(1^2, 5^2, 7^2)$ and $(7^2, 13^3, 17^2)$, leading to the equations $x^2 - 5x \pm 6 = 0$ and $x^2 - 13x \pm 30$.

There are some cases that might be found empirically, such as

$$(a, b) = (5, 6), (13, 30), (17, 60).$$

A possible family is given by

$$(a, b) = (r^2 + (r+1)^2, r(r+1)(2r+1)).$$

Another approach begins with supposing that the two roots of the quadratic $x^2 - ax - b$ are u and $a - u$ and of the quadratic $x^2 - ax + b$ are v and $a - v$. Then

$$u(a-u) + v(a-v) = 0$$

from which

$$a = \frac{u^2 + v^2}{u + v}.$$

Now it becomes a question of which integers can be written in the form $(u^2 + v^2)/(u+v)$.

As an exercise, students can be asked to explain and check that we can replace u by $a-u$ and v by $a-v$, either both at once or one at a time, without changing the value of $(u^2+v^2)/(u+v)$.

Problem 36. A *round robin tournament* is a sporting event that involves a number of competitors, either teams or individuals, in which each pair competes once. We will also suppose that the result of a match is a win (W) for one player and a loss (L) for the other.

(a) Suppose that n players are competing in a round-robin tournament, so that each player competes in $(n-1)$ matches. How many matches are there in the tournament.

(b) For each i, suppose that the ith player has w_i wins and z_i losses, for $1 \le i \le n$. Prove that

$$w_i + z_i = n - 1 \qquad \text{and} \qquad w_1 + w_2 + \cdots + w_n = z_1 + z_2 + \cdots + z_n.$$

(c) Make up some examples of results of round-robin tournaments and investigate the sums $\sum_{i=1}^{n} w_i^2$ and $\sum_{i=1}^{n} z_i^2$. Account for your observation.

Discussion. For example, suppose that we have six competitors, labelled **1, 2, 3, 4, 5, 6**. Then there will be 15 matches to be played; we can draw up a table recording the results. This table has six rows and six columns indicating the plays. Row x and Column y will intersect in a cell in which we will enter W if x defeats y and L if x is beaten by y.

A possible outcome of a tournament is given by this table:

Players	**1**	**2**	**3**	**4**	**5**	**6**	# wins	# losses
1	–	W	W	W	W	W	5	0
2	L	–	W	W	W	W	4	1
3	L	L	–	W	W	L	2	3
4	L	L	L	–	W	W	2	3
5	L	L	L	L	–	W	1	4
6	L	L	W	L	L	–	1	4

Not only is the total number of wins equal to the total number of losses:

$$5+4+2+2+1+1 = 15 = 0+1+3+3+4+4,$$

but the sums of the squares of these two sets of numbers are equal:

$$5^2+4^2+2^2+2^2+1^2+1^2 = 51 = 0^2+1^2+3^3+3^3+4^4+4^2.$$

A check of other examples indicates that the sums of the squares of the numbers of wins for the players will equal the sums of the squares of the numbers of losses.

The beauty of this example is that it is possible to explain this using tools that are accessible to a beginning Grade 9 student in algebra. The first idea is that, if we want to show that two quantities are equal, we can do this by showing that their difference is 0. The second is to use the factorization of a difference of squares:

$$m^2 - n^2 = (m+n)(m-n).$$

The algebra is straightforward:

$$\sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n}(w_i^2 - z_i^2) = \sum_{i=1}^{n}(w_i + z_i)(w_i - z_i)$$
$$= (n-1)\sum_{i=1}^{n}(w_i - z_i) = (n-1)[(\sum_{i=1}^{n} w_i) - (\sum_{i=1}^{n} z_i)] = 0.$$

EXPOSITORY SITUATIONS

A classroom session can be devoted to analyzing the solution to a problem that students would probably not be able to solve on their own. However, the mathematics involved in the solution may be at their level and it can be insructive to take them through the solution.

Problem 1. Students are asked to write down any ten positive integers not exceeding 100. They are then told that they can find two disjoint sets within the ten for which the sum of the numbers is one is equal to the sum of the numbers in the other. (No one of the selected number is in the two sets, and some of the ten choices may not be selected for either set.) This statement is often received with some skepticism.

If a student selects the same number twice, then the two occurrences of the number make up the two sets and the statement is true. So we can assume that all the numbers chosen are distinct.

It is usually a chore to locate the two sets, but there is a simple, albeit tedious algorithm, to find them. There are $1023 = 2^{10} - 1$ possible sets (for each of the ten numbers, you have two choices, include or leave it out of the set; we subtract 1 to exclude the situation in which no number is chosen).

We may have go throught the possibilities systematically to find the sets, but it is a different matter if we just want to demonstrate the existence of the sets without having to produce them. The key is to put two significant facts together: we have more than 1000 possible sets, but the sum of the number in any one of them does not exceed $10 \times 100 = 1000$. So for each set, we calculate it sum. Since there are more sets than sums, some sum must occur twice.

Suppose we have two different sets with the same sum. It may happen that the two sets overlap. We just remove the numbers common to the two sets to get the disjoint sets we are looking for.

Problem 2: The sum of the first n cubes. Sometime during secondary school, students will comes across the remarkable fact that the sum of the first n integer cubes is equal to the square of the sum of the first n integers:

$$1^3 + 2^3 + \cdots + (n-1)^2 + n^3 = (1 + 2 + \cdots + (n-1) + n)^2.$$

For example, $1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 = (1 + 2 + 3 + 4)^2$ and $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 100 + 125 = 225 = 15^2 = (1 + 2 + 3 + 4 + 5)^2$. This is often proved by an induction argument, which may not deliver understanding of why the result *has* to be true, rather than just *happening* to be true.

Discussion. A prototypical case, taking sums up to 5, will help us see what is going on. Begin with the upper left corner of the multiplication table:

1	*2*	**3**	*4*	**5**
2	*4*	**6**	*8*	**10**
3	**6**	**9**	*12*	**15**
4	*8*	*12*	*16*	**20**
5	**10**	**15**	**20**	**25**

The sum of the numbers in the first row is 15. The numbers in the second row are double those in the first, so their sum is 2×15. Likewise, the sum of the numbers in the third row are 3×15, in the fourth row, 4×15, and in the fifth row, 5×15. So all the numbers in the table add up to $(1+2+3+4+5) \times 15 = 15 \times 15 = 15^2$.

We can look at the sum of the numbers in another way. I have partitioned the table into *gnomons* or carpenter's squares. The first gnomon consists of 1; the second of all the multiples of 2 up to 4; the third all the multiples of 3 up to 9; the fourth multiples of 4 up to 16; the fifth all the multiples of 5 up to 25.

Now add the numbers in each gnomon:

$$1; \qquad 2 + 4 + 2 = 8; \qquad 3 + 6 + 9 + 6 + 3 = 9 + 9 + 9 = 3 \times 9 = 3^3;$$

$$\begin{aligned} 4 + 8 + 12 + 16 + 12 + 8 + 4 &= (4 + 12) + (8 + 8) + (12 + 4) + 16 \\ &= 16 + 16 + 16 + 16 = 4 \times 16 = 4^3; \end{aligned}$$

$$\begin{aligned} &5 + 10 + 15 + 20 + 25 + 20 + 15 + 10 + 5 \\ &\qquad = (5 + 20) + (10 + 15) + (15 + 10) + (20 + 5) + 25 \\ &\qquad = 5 \times 25 = 5^3. \end{aligned}$$

The sum of all the number in the table is thus

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3,$$

which we have already seen to be equal to $(1 + 2 + 3 + 4 + 5)^2$.

Extension. There are remarkably many finite sequences of integers, not necessarily all distinct, for which the sum of their cubes is equal to the square of their sum. It is easy to list all the cases when there are three, four, or five positive integers (when some are allowed to be negative, then there are many more possibilities).

Here is an illustration of how to find some other sets. Take any positive integer, say 24. Write down all of its positive divisors: 1, 2, 3, 4, 6, 12, 24. For each divisor, write down the number of all its positive divisors. 1 has one divisor; 2, 3 have two; 4 has three; 6 and 8 have four; 12 has six and 24 has eight. List these numbers (including repeats): 1, 2, 2, 3, 4, 4, 6, 8. We find that

$$\begin{aligned} 1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 4^3 + 6^3 + 8^3 &= 900 \\ &= (1 + 2 + 2 + 3 + 4 + 4 + 6 + 8)^2. \end{aligned}$$

Problem. The odd primes are $3, 5, 7, 11, 13, 17, 23, 29, \ldots$. Prove that the sum of two consecutive odd primes is the product of three positive integers bigger than 1. For example, $5 + 7 = 12 = 2 \times 2 \times 3$; $17 + 23 = 40 = 2 \times 4 \times 5$.

Discussion. There are three ingredients that are involve in the solution: the sum of two odd numbers is even; the average of two numbers is their sum divided by 2; the average of two numbers lies between them. What is the significance of the two primes being *consecutive*?

Problem 3. Let p be a prime number other than 2 or 5. Show that there is a number, all of whose digits are 9, which is divisible by p. Let such a number with the fewest digits by n and look at the multiples of n/p. What do you observe?

Discussion. We have $9 \div 3 = 3$, $999999 \div 7 = 142857$, $999999 \div 13 = 076923$, $999 \div 37 = 027$. To see that we always get an exact division, imagine dividing each of the numbers 9, 99, 999, $\ldots$ by p and look at the remainders. There are only p possible remainders, so eventually we will come across one we have seen before. Suppose that $a > b$ and the number with a nines, namely $10^a - 1$ leaves the same remainder as the number with b nines, namely $10^b - 1$. Then their difference $10^b(10^{a-b} - 1)$ must be divisible by p.

To understand the phenomenon you observe with the multiples of n/p use the fact that $k \times (n/p) = (kn)/p$ and the fact that $2 \times 999\ldots99 = 1999\ldots998$, $3 \times 999\ldots997$, *etc.*.

Problem 4. Let $A = 4444^{4444}$, B be the sum of the digits of A, C be the sum of the digits of B and D be the sum of the digits of C. What is D?

Discussion. This is an Olympiad problem, but while it takes the insight of a strong competitor to see the way forward, the strategy and details of a solution are within the grasp of a secondary student and involve some useful issues.

When competitors see "the sum of the digits", they might think of "casting out nines", which is the principle that the remainder of any positive integer when divided by 9 is the same as the remainder of the sum of its digits. This is probably not regularly taught in schools as a checking mechanism as it was a few generations ago, but it is straightforward to explain and is knowledge that is still worth having.

We define the *digital sum* of a positive integer is that single digit number obtained by adding its digits, and continue adding the digits of the results until you get down to the single digit. The digital sum of the sum (resp. product) or two numbers is equal to the digital sum (resp. product) of the sum (resp. product) of their digital sums.

In particular, the digital sums of A, B, C, D are equal.

The second strand in the solution is to observe that when you add the digits of a number you get a smaller number (much smaller when the original number is large). So it might be fruitful to get some idea of how big D can be.

Let us begin. The digital sum of 4444 is 7, so the digital sum of A is equal to the digital sum of 7^{4444}. Since $7^2 = 49$ and $7^3 = 343$, we see that the digital sum of 7^3 is 1. Since $7^{4444} = (7^3)^{1481} \times 7$, the digital sum of A is 7. Thus, D leaves a remainder of 7 upon division by 9.

How big are A, B, C, D? Since B cannot be bigger than the number of digits in A multiplied by 9, we need to get a handle of how long anumber A is. Since every number less than 10 has one digit, less than 10^2 has at most two digits, less than 10^3 has at most three digits, and so on, we look for a power of 10 that exceeds A.

Since $4444 < 10^4$, $A < (10^4)^{4444} < 10^{20000}$, so A has fewer than 20000 digits and this makes $B < 180000$. Then $C < 45$ and $D < 13$. Thus D is a number less than 13 that leaves a remainder of 7 upon division by 9; there is one possibility.

Problem 5: The 7-11 Problem. A man goes into a convenience store and purchases four items. The clerk performs some arithmetic on his calculator and announces that the cost is \$7.11. However, the customer noticed that by mistake, the clerk punched the times rather than the plus button, and so multiplied the four prices. The clerk apologized and redid the calculation, correctly this time. “That will be \$7.11, please!”. What did the four items cost?

Discussion. This problem, named in honour of a chain of stores (presumably open between 7am and 11pm), has an enticing air of mystery about it. The first thing is how to deal with the decimal point, and one way to handle the difficulty is to express all the prices in cents. Let the four prices be a, b, c, d. Then we are given that

$$\frac{a}{100} \times \frac{b}{100} \times \frac{c}{100} \times \frac{d}{100} = \frac{711}{100}.$$

whereupon $abcd = 711 \times 10^6 = 3^2 \times 79 \times 2^6 \times 5^6$.

Since 79 is prime, one of the numbers, say a must be a multiple of 79, *i.e.* 79, 158, 237, 316, 395, 474, 553, 632. Since it is not possible for all four numbers to be multiples of 5, one can argue that at least one of them is divisible by $125 = 5^3$, so that one of the numbers must be one of 125, 250, 375, 500, 625. Can two of them be divisible by 125?

There is only one answer: $(1.20, 1.25, 1.50, 3.16)$. While this can be verified by a calculator, a more efficient way to do it is to convert the numbers to vulgar fractions and use cancellations to find the product:

$$\left(\frac{6}{5}\right) \times \left(\frac{5}{4}\right) \times \left(\frac{3}{2}\right) \times \left(\frac{79}{25}\right) = \frac{3^2 \times 79}{4 \times 25}.$$

Problem 6: Products of consecutive terms of a quadratic sequence. There are a number of situations in which the product of two consecutive terms in a sequence of integers is also in the sequence. This obviously happens in a sequence of squares. But it also happens in other sequences as well:

(1) $(n^2 + 1 : 0, 1, 5, 10, 17, 26, 37, 50, 65, 82, 101, 122, 145, 170, \dots)$;

(2) $(n^2 - 1 : -1, 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 168, \dots)$;

(3) $(n^2 + n + 1 : 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, \dots)$;

(4) Oblong numbers: $(n(n + 1) : 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, \dots)$.

For each of the sequences, what can you say about the location of the product of two consecutive terms. Can you find a formula?

Discussion. This turns out to be quite a rich territory to explore. At a basic level, it involves students taking the form of the terms of the sequence as an algebraic expression and showing that product of consecutive terms are of the same form. For example, in the case of the oblong numbers, can you write $[n(n+1)][(n+1)(n+2)]$ in the form $m(m+1)$ for some m that depends on n? In this case, it may be more convenient to write the product of consecutives as $[(n-1)n][n(n+1)]$. This allows for practice in algebraic judgment and manipulative skills.

There is a more general result lurking here. Let $p(n) = n^2+bn+c$ be a quadratic polynomial whose leading coefficient is 1. Then, for each integer n, $p(n)p(n+1) = p(m)$ for some integer m. There are several ways of establishing this, but the easiest approach is to write $q(k) = p(n + k) = k^2 + uk + v$, where $u = 2n + b$ and $v = n^2 + bn + c = p(n)$. This is a quadratic in k whose leading coefficient is 1, so the problem reduces to showing that $q(0)q(1) = q(m)$ for some integer m.

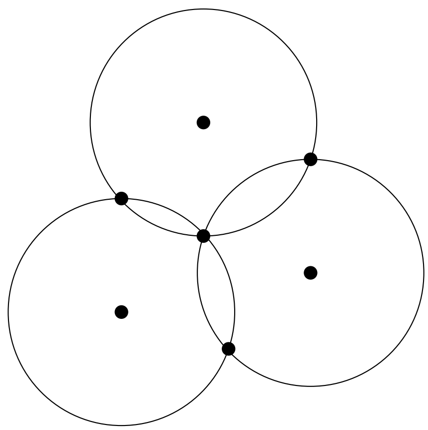
But

$$\begin{aligned} q(0)q(1) &= v(1 + u + v) = v^2 + uv + v \\ &= q(v) = p(n + v) = p(n + p(n)). \end{aligned}$$

The key to this is to recognize that the expression $v(1+u+v)$ can be read in two ways, as the product from which it is derived, and as the value of the polynomial $q(k)$ when $k = v$.

When he was still in high school, James Rickards from Ottawa was exposed to this situation. He noted that $p(n)p(n+1)$ is a quartic polynomial in n two of whose roots have the same sum as the other two, and that the right side $p(n + p(n))$ is the composite of two quadratic polynomials. He was able to generalize this, first to show that any quartic polynomial can be written as the composite of two quadratics if and only if the sum of one pair of roots is equal to the sum of the others, and more to show a more complicated requirement for a polynomial of any degree to be the composite of two polynomials of lower degree. His paper can be found at *American Mathematical Monthly* 118:4 (April, 2011), 358-363.

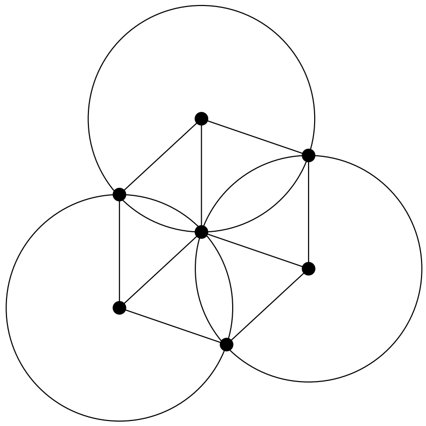
Problem 7: The beer mug problem. You are given three circles whose radii are equal and whose circumferences have a common point. There are three other points where the circumferences intersect in pairs.



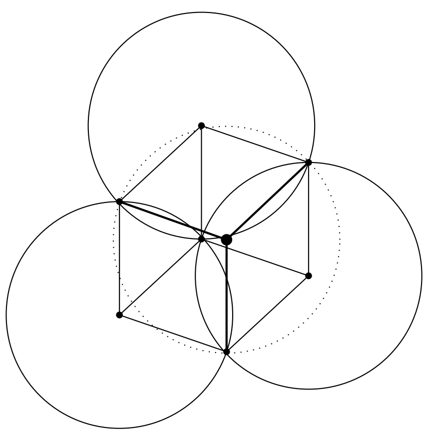
The problem is to show that these last three points lie on the circumference of a circle exactly the same size as each of the three given circle.

Discussion. The name of the problem comes from its formulation in terms of a sweating beer mug placed three times, just so, on a bar, so that the moist rings they leave intersect in a single point. The mug can be placed a fourth time so that its circumference passes through the pairwise intersection points of the three rings. You can try this at home. In our present sensitive age, it might be that the title of the problem is not suitable for a young audience.

It is easy to tie yourself in knots trying to solve this in a technical way. However, the path forward becomes visible if you join all pairs of points which you know to be radii of one of the circles. This gives you what appear to be nine edges of the skeleton of a cube. It is as though you are looking down the cube from above and are able to see seven of its eight vertices.



The trick is to complete the representation of the cube to show the eighth vertex. It turns out that this is the location of the centre of the circle whose circumference passes through the intersection points of the pairs of the circles. Briefly, to check this circle has the same size is to look at the quadrilaterals whose sides are bounded by radii of the circles; these quadrilaterals, having all sides equal, are rhombi and so their opposite sides are parallel. When we determine the position of the eighth vertex of the cube, it is the fourth point of a rhombus two of whose sides are radii and whose opposite sides are parallel.



Problem 8: a bit of history. The Greek mathematician, Diophantus (third century AD) wrote five books in which he discussed problems of an algebraic nature. His solution to Problem 20 in Book 4 is ingenious. You are asked to find four vulgar fractions such that the product of any two of them plus 1 is the square of a fraction.

He begins with the observations:

$$(x+1)^2 = x(x+2)+1;$$
$$(2x+1)^2 = x(4x+4)+1;$$
$$(3x+1)^2 = x(9x+6)+1.$$

If we start withany x and let the four numbers be x, $x+2$, $4x+4$ and $9x+6$, then we are partway home. It turns out that

$$(x+2)(4x+4)+1=(2x+3)^2;$$
$$(4x+4)(9x+6)+1=(6x+5)^2.$$

It remains only to deal with $(x+2)(9x+6)=9x^2+24x+13$. Fortuitously, the coefficient 9 is square, so we can compare it to a square of the form $(3x+t)^2$ for some value of t and wind up with a linear equation in x. Diophantus' choice was $t=-4$. Setting

$$9x^2+24x+13=(3x-4)^2=9x^2-24x+16$$

leads to $48x=3$ and $x=1/16$. The sought-after quadruple is

$$\left(\frac{1}{16},\frac{33}{16},\frac{17}{4},\frac{105}{16}\right).$$

Extension. There are a great many triples (x,y,z) of integers for which $xy+1$, $xz+1$, $yz+1$ are all squares. Many of them can be extended to quadruples. This can be investigated with 1 replaced by other integers. Suffice it to say that this is an extremely rich area of investigation for any student interested in the vagaries of numbers. And it is a situation where Fibonacci numbers, Pell's equation and pythagorean triples figure in a big way.

Problem 9. The number 142857 has the interesting property that its first six multiples (142857, 284714, 428571, 571428, 714285, 857142) have the same digits in some cyclic order. The mystery is partly uncovered when we find that $7\times 142857=999999$.

It turns out that if we have a prime other than 2 and 5 (why), there is a number all of whose digits are 9 that is divisible by 9. For example, $3\times 3=9$, $11\times 9=99$, $13\times 76923=999999$, $37\times 27=999$. What is the situation with 17, 19, 23, 29, 31; a calculator or computer may be necessary here.

To see that for a given prime p, there is a number whose digits are all nines involves a simple, but very powerful principle that can be applied in many different situations: *the pigeonhole principle.* If you have a number of letters to be distributed into a number of pigeonholes, and the number of letters exceeds the number of pigeonholes, then some pigeonhole must receive at least two letters.

Let us apply that here: look at the numbers 9, 99, 999, *etc* in turn; divide them by p and record the remainder, which will be some number between 1 and $p-1$, inclusive. At some point, the same remainder will appear twice. This means that the difference between the two dividends will be divisible by p. In algebraic terms, the numbers we are looking at have the form 10^m-1. If $m>n$ and 10^m-1 and 10^n-1 have the same remainder, then p divides $(10^m-1)-(10^n-1)=(10^{m-n}-1)10^n$. It cannot divide 10^n which is divisible only by the primes 2 and 5, so $10^{m-n}-1$ (with $m-n$ 9s) is divisible by p.

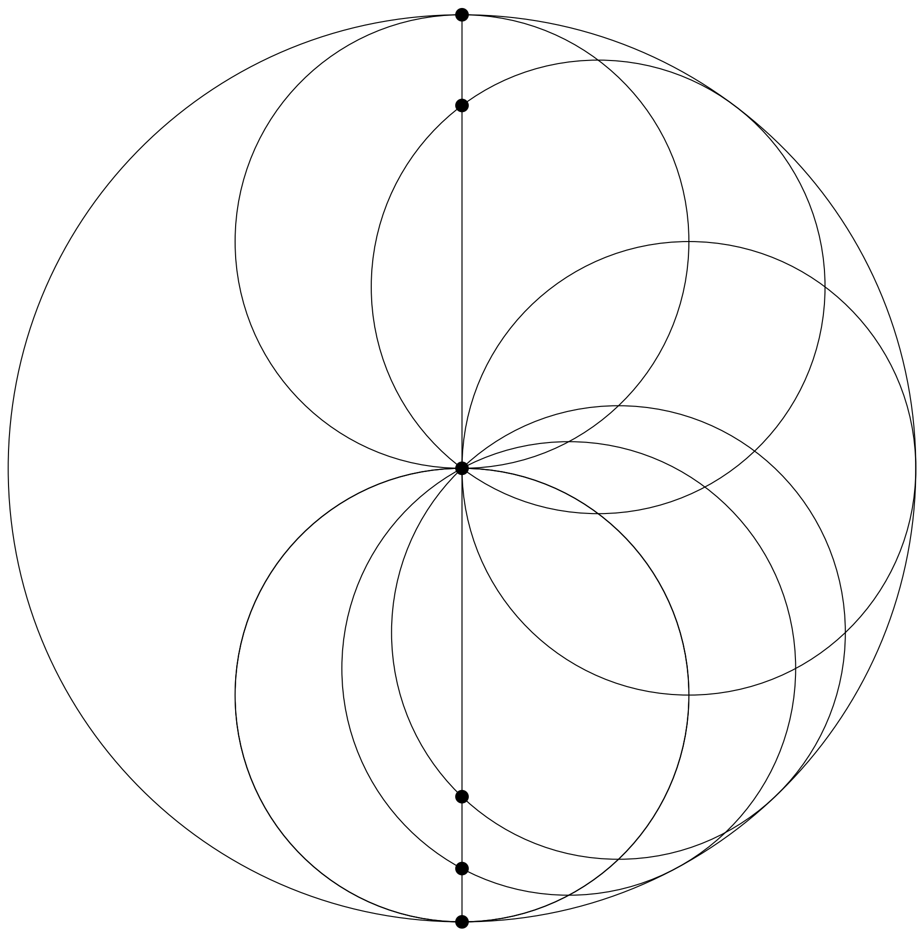
Now have the pupils take a given prime p and divide it into the smallest number whose digits are all 9 and look at the multiples of the quotient. What do they observe?

Problem 10. Find all positive integers with this property: any positive integer less than it and greater than 1, which has no common divisor with it must be prime.

Finding the first few is just an exercise: 2, 3, 4, 6, 8, 12, 18, 24, 30. The key is to note that once we get past the square of a prime, then that prime must divide a number with the property. Establishing that we have the complete list is a harder problem that the pupils are not likely to solve, but will have a good sense of what is going on and what needs to be looked at.

Problem 11. When my children were young, we paid a family visit to the science museum in Ottawa. My attention was caught be an exhibit about pistons of different designs. One of them was an application of a geometric result. Suppose that we have a large circle, along the inner cricumference of which rolls a smaller circle of half the diameter. The path of a fixed point on the circumference of the smaller circle travels back and forth along a diameter of the larger circle.

In the diagram below, we show a number of positions of the inner circle as it rolls inside the outer circle. We start with the inner circle touching the larger circle at the top. As it rolls to the right, the inner circle rotates counter-clockwise and the fixed point indicated by a dot travels down the vertical tangent until it reaches at lowest position. As it continues to the right the dot travels back up the diameter.



To design a piston, we attach a vertical rod to the fixed point on the inner circle. We then force the inner circle to rotate around the inside of the outer one, driving the piston up and down.

Discussion. How many times does the inner circle rotate as it makes one circuit around the centre of the larger? Does this give any insight why the locus of the fixed point is a diameter? How does this depend on the ratio of the two radii? What happens if the second circle travels on the outside of the circumference of the first?