Quiver variety components and minuscule combinatorics

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Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of trace 0 matrices and $V = \mathbb{C}^n$. The standard basis vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ form a basis for V that has several favorable properties:

1. Each basis vector is an eigenvector for the action of the subalgebra $\mathfrak{h}$ of diagonal matrices, i.e. $\mathrm{diag}(t_1, \ldots, t_n) \cdot \mathbf{v_k} = t_k \mathbf{v_k}$
2. The matrices $E_{i,j} = (e_{mn})$ s.t. $e_{mn} = \begin{cases} 1 & \text{if } (m,n) = (i,j) \\ 0 & \text{else} \end{cases}$ for $i \neq j$ "almost permute" these vectors, i.e. $E_{i,j} \cdot \mathbf{v_j} = \mathbf{v_i}$ and $E_{i,j} \cdot \mathbf{v_k} = \mathbf{0}$ for $k \neq j$.
3. In fact we only need to use the matrices $F_i = E_{i+1,i}$ to reach any basis vector from $\mathbf{v_1}$.

Thus we can encode the representation as a colored directed graph, for example, $\mathfrak{sl}_3$ acting on $\mathbb{C}^3$ could be represented like this:

$$\mathbf{v_1} \xrightarrow{F_1} \mathbf{v_2} \xrightarrow{F_2} \mathbf{v_3}$$

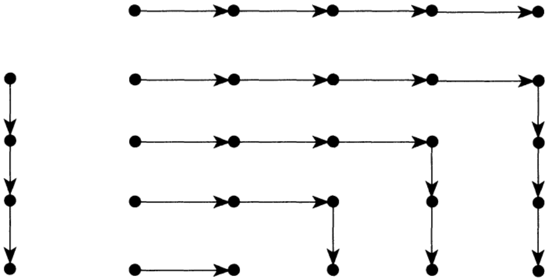
Our aim is to generalize this idea and we'd hope that the nice basis we found is compatible with tensor product decompositions and branching.

This works only as long as each weight space is one-dimensional. We already run into trouble with the adjoint representation of $\mathfrak{sl}_3$, as $\ker F_1, \ker F_2, \operatorname{im} F_1, \operatorname{im} F_2$ are all different subspaces of $\mathfrak{h}$.

$$\begin{tikzcd}
 & E_{13} \arrow[dl, "F_2"'] \arrow[dr, "F_1"] & \\
E_{12} \arrow[dr, "F_1"'] & & E_{23} \arrow[dl, "F_2"] \\
 & \mathfrak{h} = ? \oplus ? \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
E_{21} \arrow[dr, "F_2"'] & & E_{32} \arrow[dl, "F_1"] \\
 & E_{31} &
\end{tikzcd}$$

Fortunately there is a way of fixing this problem, due to Kashiwara [Kas91] , by going first to the quantized universal enveloping algebra $U_q(\mathfrak{g})$ and then taking a limit as $q \to 0$ in a suitable sense. It turns out that in this setting, choosing a good basis is always possible. This object, the directed with vertices the basis elements and the edges labeled by the action of the lowering operators is called a **crystal**. Since the representation theory of $U_q(\mathfrak{g})$ is very similar to that of $U(\mathfrak{g})$, we can use this combinatorial gadget to study representations.

What is the benefit of crystals? Combinatorics. For $\mathfrak{g} = \mathfrak{sl}_2$-crystals, tensor product decompositions are given by:



We know that for an irreducible $\mathfrak{sl}_n$-representation V_λ of highest weight λ, $\dim(V_\lambda) = \#SSYT(\lambda)$ with entries up to n. The crystal of the adjoint representation of $\mathfrak{sl}_3$ is

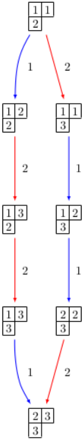


Figure: The crystal $B(\omega_1 + \omega_2)$ for A_2

Let $\mathfrak{g}$ be a simply laced Lie algebra and let ω_p be a minuscule fundamental weight. Consider the the poset Δ_+ of positive roots of $\mathfrak{g}$ and the subposet Δ_+^p of roots that lie above α_p. For type A_3, the poset Δ_+^2 is:

$$\begin{tikzcd}
& & \alpha_1+\alpha_2+\alpha_3 \arrow[dl, dash] \arrow[dr, dash] & & \\
& \alpha_1+\alpha_2 \arrow[dl, dash] \arrow[dr, dash] & & \alpha_2+\alpha_3 \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha_1 & & \alpha_2 & & \alpha_3
\end{tikzcd}$$

Let $B(k\omega_p)$ be the crystal of the $\mathfrak{g}$ irrep with highest weight $k\omega_p$. Then the set of order-preserving maps

$$\{\phi : \Delta_+^p \to \{0, 1, \ldots, k\}\}$$

(called **reverse plane partitions**) is a model for the crystal $B(k\omega_p)$. Rpps are a generalization of Gelfand-Tsetlin patterns. For type A they are in bijection with semistandard tableaux of shape $p \times k$, where column i of the rpp is the shape of the tableaux with entries $\leq i$.

Example 1

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \leftrightarrow \left(\begin{array}{ccccccc} 3 & & 3 & & 0 & & 0 \\ & 3 & & 2 & & 0 & \\ & & 3 & & 1 & & \\ & & & 2 & & & \end{array} \right) \xrightarrow{\mathrm{refl} \circ \mathrm{rot}^{90^\circ}} \left(\begin{array}{ccc} & 1 & \\ 2 & & 2 \\ & 3 & \end{array} \right)$$

The lowering operator f_i acts on rpps by decreasing an entry on the i-th column. For example, for $B(k\omega_2)$ in type A_3, we have

$$f_1 \begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{pmatrix} & a & \\ b-1 & & c \\ & d & \end{pmatrix}$$

$$f_2 \begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{cases} \begin{pmatrix} & a-1 & \\ b & & c \\ & d & \end{pmatrix} & \text{if } a+d \leq b+c \\ \begin{pmatrix} & a & \\ b & & c \\ & d-1 & \end{pmatrix} & \text{if } a+d > b+c \end{cases}$$

$$f_3 \begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{pmatrix} & a & \\ b & & c-1 \\ & d & \end{pmatrix}$$

(if the resulting array is an rpp)

Let Q be an orientation of $\mathfrak{g}$'s Dynkin diagram with vertex set I, and Q^* be the opposite orientation. Consider the doubled quiver $\overline{Q} = Q \cup Q^*$. Let $\mathbb{C}\overline{Q}$ be the path algebra of $\overline{Q}$. Consider the element

$$\rho = \sum_{e \in E(\overline{Q})} \varepsilon(e) e^* e$$

where $\varepsilon(e) = 1$ if $e \in E(Q)$ and -1 if $e \in E(Q^*)$. The algebra $P(Q) = \mathbb{C}(\overline{Q})/(\rho)$ is called the **preprojective algebra of Q.**

There is a module $P(p)$ of the preprojective algebra with basis indexed by Δ_+^p

$$\begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[dr] & \\ \mathbb{C} \arrow[dr] & & \mathbb{C} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 3 \end{array}$$

Figure: $P(2)$ for A_3

We are interested in the space $L(k\omega_p) = \{M \subseteq P(p)^{\oplus k}\}$ of submodules.

$L(k\omega_p)$ has connected components indexed by possible dimension vectors of M.

Conjecture 2

The irreducible components of $L(k\omega_p)$ are indexed by reverse plane partitions.

For example, consider $L(2\omega_2)$ in type A_3. Let M be a submodule with dimension vector $(1, 2, 1)$. Write M_i for the subspace corresponding to the i-th node of the Dynkin diagram. To choose M_2, we have to choose a 2-dimensional subspace of $\mathbb{C}^2 \oplus \mathbb{C}^2$ stable under the linear map

$$T(x, y, z, w) = (0, 0, x, y).$$

1. If $M_2 = \ker T$, and we can choose M_1 and M_3 arbitrarily, this corresponds to the rpp

$$\begin{pmatrix} & 0 & \\ 1 & & 1 \\ & 2 & \end{pmatrix}$$

2. If $M_2 \neq \ker T$, then M_1 and M_3 are determined, this corresponds to the rpp

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

Conjecture 3

The lowering operator f_i on the rpps corresponds to taking a generic submodule with quotient the simple module S_i (the 1-dimensional module supported at vertex i).

Consider the case $L(k\omega_2)$ in type A_3. Let $\phi = \begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix}$ be an rpp, and let M be a module in the component indexed by ϕ. For simplicity, we identify the subspaces $M_1 = B$ and $M_3 = C$ with their images in $M_2 = A + D$. Then M looks like this:

$$\begin{tikzcd} & A \arrow[dl, "T"'] \arrow[dr, "T"] & \\ B \arrow[dr, "id"'] & & C \arrow[dl, "id"] \\ & D & \end{tikzcd}$$

If we are looking for a submodule of M that fits into the SES

$$0 \to f_2(M) \to M \to S_2 \to 0$$

then we have to choose an $a+d-1$-dimensional subspace of $A+D$. To be a submodule of M, this subspace needs to contain B and C. Generically, this subspace will not contain all of D, unless $B+C=D$, in which case we are forced to contain all of D.
B and C are two subspaces of D and both contain $T(A)$. Therefore generically $\dim(B+C)=b+c-a$, so we have $B+C=D \Leftrightarrow b+c-a \geq d$, or equivalently, if

$$a+d \leq b+c.$$

and we see that this is the same rule as the lowering operator on the rpps (10).

[Kas91] M. Kashiwara.
On crystal bases of the Q-analogue of universal enveloping algebras.
Duke Math. J., 63(2):465–516, 1991.