Tutorial 9
Recurring Reality

Learning Objectives

> In this tutorial you will construct and solve recurrences to compare closed forms.
>
> These problems relate to the following course learning objectives: *describe solutions to iterated processes by relating recurrences to induction, generating functions, or combinatorial identities,* and *identify when an exact solution is intractable, and use estimates to describe its approximate size.*

Counting Stories

You are managing **Recurring Reality**, a store that specializes in interactive adventure virtual reality. A customer puts on a VR headset and selects either a 10, 20 or 30 minute adventure. There is one type of 10 minute adventure, two types of 20, and three types of 30. Each adventure can be continued as many times as a customer wants, or after it ends they can choose different one, or leave. Choosing the same adventures in a different order gives a different experience.

1. Give a recurrence and initial conditions for $RR(n)$, the number of possible experiences a customer can have in $10n$ minutes.

2. Determine the number of different experiences that a customer can have in any increment of ten minutes, up to one hour.

Like every popular store in Toronto, you soon have some imitators trying to compete with you. **Big Show Notation** offers two types of 10 minute adventures and nothing else. **Generating Fun** offers one type of 10 minute adventure and three types of 20 minute adventures. We will use $BSN(n)$ and $GF(n)$ for the number of possible experiences a customer can have after $10n$ minutes.

3. Based on having as many or more experiences for every time up to 30 minutes, Big Show Notation is claiming that they give customers more experiences than Recurring Reality. For which amounts of time is this true?

4. Show that Generating Fun has more experiences than Recurring Reality for 20 and 40 minutes. Find a closed form for $GF(n)$.

5. Is $RR(n) = O(GF(n))$? Is $GF(n) = O(RR(n))$?

Improving Experiences

A new story installer can change the types of adventures you have installed. Your goal is to maximize the number of distinct experiences for your customers, to continue to appear mathematically better than your competition.

6. You can remove all of your 30 minute adventures to add one extra 20 minute adventure and one 40 minute adventure. Should you do it?

7. You have the option to convert a 30 minute adventure into two 40 minutes adventures. Should you do it?

8. At least how many 30 minute adventures should you request in exchange for removing all of your 20 minute adventures?

1. $RR(n) = RR(n-1) + 2RR(n-2) + 3RR(n-3)$, with $RR(0) = RR(1) = 1$, $RR(2) = 3$ (either the ten minute adventure twice, or one of the two 20 minute adventures).

2. The number of experiences, starting with 0 minutes, are: $[1, 1, 3, 8, 17, 42, 100]$, using the previous recurrence.

3. This is only true for 10 and 20 minutes. Note that $BSN(n) = 2^n$. We can show that $RR(n) > 2^n$ for all $n \geq 4$, by using the base cases $RR(4)$, $RR(5)$, $RR(6)$, and strong induction. Our hypothesis should include three previous terms, or all terms ≥ 4 - it is not true for every n.

4. $GF(n) = GF(n-1) + 3GF(n-2)$, with $GF(0) = GF(1) = 1$. Hence $GF(2) = 4 > RR(2)$, and $GF(4) = 19 > RR(4)$. We get $GF(n) = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{13}}$, where $\alpha = \dfrac{1+\sqrt{13}}{2}$, $\beta = \dfrac{1-\sqrt{13}}{2}$.

5. We have $\alpha > |\beta|$, so $GF(n) = O(\alpha^n)$, and $\alpha^2 = \alpha + 3$. A closed form for $RR(n)$ has roots that satisfy $x^3 - x^2 - 2x = 3$. If every root is less than α in absolute value, then $\alpha^3 - \alpha^2 - 2\alpha > 3$. The left side simplifies to α, which we can check is less than 3. So $RR(n)$ grows faster than $GF(n)$. We can also use a CAS to compute the roots (there is one real root near 2.37, and two complex roots with absolute values less than 1). Thus $GF(n)$ is $O(RR(n))$, but not the other way around.

6. No. This will give a recurrence whose roots satisfy $x^4 = x^3 + 3x^2 + 1$, which has two real roots (near 2.35 and -1.43) and two complex roots of absolute values less than 1. This will grow like 2.35^n, which is slightly less than what we started with.

7. No. This will give a recurrence whose roots satisfy $x^4 = x^3 + 2x^2 + 2x + 2$, which has two real roots (near 2.36 and at -1) and two complex roots of absolute values less than 1. This will grow like 2.36^n, which is slightly less than what we started with.

8. We should get at least 8 to increase the number of experiences in the long run. We should also compare the number of experiences for small numbers to see when the new version gives more experiences. It starts to give more at 220 minutes for 8; at 90 minutes for 9; and at 60 minutes for 10.