Tutorial 8

Generating Functions

Learning Objectives

> In this tutorial you will practise manipulating generating functions - moving between sequences, closed forms of generating functions, and formulas for coefficients.
>
> These problems relate to the following course learning objectives: *describe solutions to iterated processes by relating recurrences to induction, generating functions, or combinatorial identities.*

Identifying Functions

1. Match the descriptions of sequences to the initial terms in the sequence and the closed form of the generating function.

   (a) The alternating sign sequence, $[(-1)^n]$.

   (b) The sequence of squares, $[n^2]$.

   (c) The sequence $a_n = \begin{cases} 3 \text{ if } n \text{ is a multiple of } 3 \\ 1 \text{ otherwise.} \end{cases}$

   (d) The powers of 3, $[3^n]$.

   (e) The natural numbers, $[n]$.

$[3, 1, 1, 3, 1, 1, 3, 1, \ldots]$	______	$\dfrac{1}{(1-3x)}$	______
$[1, 3, 9, 27, 81, \ldots]$	______	$\dfrac{1}{1-x} + \dfrac{2}{1-x^3}$	______
$[0, 1, 4, 9, 16, 25, 36, \ldots]$	______	$\dfrac{1}{(1-x)} - \dfrac{2x}{(1-x^2)}$	______
$[0, 1, 2, 3, 4, 5, 6, \ldots]$	______	$\dfrac{x}{(1-x)^2}$	______
$[1, -1, 1, -1, 1, -1, 1, -1, \ldots]$	______	$\dfrac{2x}{(1-x)^3} - \dfrac{x}{(1-x)^2}$	______

2. Suppose $a_n = Bn + C$ is a linear sequence. Show that it has generating function $\dfrac{r}{(1-x)} + \dfrac{t}{(1-x)^2}$ for some r, t depending on B and C.

3. Show that any polynomial sequence $p(n)$ has a generating function

$$\sum_{i=1}^{d+1} \frac{c_i}{(1-x)^i}$$

   for some constants c_i, where d is the degree of p.

1. The sequences are: (c), (d), (b), (e), (a). The generating functions are: (d), (c), (a), (e), (b). These can be built from the geometric series

$$1 + ax + a^2x^2 + a^3x^3 + \cdots = \frac{1}{1-ax}$$

   and the formal derivatives when $a = 1$,

$$1+2x+3x^2+4x^3+5x^4\cdots = \frac{1}{(1-x)^2} \qquad , \qquad 2+6x+12x^2+20x^3+\cdots = \frac{2}{(1-x)^3}.$$

   The coefficients of x^n in the first series are $(n+1)$, and in the second series are $(n+1)(n+2)$.

2. Taking $t = B$ and $r = C - B$, we can sum the two series $\dfrac{r}{(1-x)} + \dfrac{t}{(1-x)^2}$ to give the coefficients $Bn + C$.

3. The coefficients of x^n in $\dfrac{(k-1)!}{(1-x)^k}$ are $P(n+k-1, k-1)$, either by taking derivatives k times, or by comparing to a stars and bars distribution. Hence, they are all polynomials of degree $k-1$ in the variable n, so taking the set of all of them of degree 0 to d gives a spanning set of the vector space of polynomials of degree up to d.