Tutorial 7

Latin Squares

Learning Objectives

> In this tutorial you will be estimating the number of Latin squares of order n. A *Latin square* is an $n \times n$ arrangement using each number $\{1, 2, \ldots, n\}$ once in each row and column (like a Sudoku).
>
> These problems relate to the following course learning objectives: *Identify when an exact solution is intractable, and use estimates to describe its approximate size*, and *describe solutions to iterated processes by relating recurrences to induction and combinatorial identities*.

Small Latin Squares

There are only 2 Latin squares of size 2×2:

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \quad , \quad \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$$

1. Find all 3×3 Latin squares.

Suppose the first row of a Latin square is $(12 \ldots n)$. Then every other row is a derangement of $[n]$.

2. Show that the following partially filled square can be completed to a Latin square in 4 ways.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & & & \\ 3 & & & \\ 4 & & & \end{matrix}$$

3. Using this, show that there are $(4!)^2$ Latin squares of size 4×4.
4. After choosing two derangements for rows 2 and 3, the last row is determined. Draw a graph where the vertices are derangements of $[4]$, and there is an edge between any two vertices if they can be used as rows 2 and 3 together. How many triangles does this graph have?

Estimates

5. Show that the number of Latin squares of size $n \times n$ is at most $n!^n$.
6. Suppose $n - k$ rows have been filled in. Let S_i be the set of elements that have not yet appeared in column i. Show by induction on the size of S_i that the number of ways to fill in the next row is $k!$. [Hint: consider the choice for the last column].
7. Show that the number of Latin squares is at least $n!(n-1)! \cdots 2!1!$.
8. How do the upper and lower bounds compare?

1. There are 12, by reordering the rows of:

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \quad , \quad \begin{matrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{matrix}.$$

2. We have

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} , \quad \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} , \quad \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{matrix} , \quad \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix} .$$

3. Since there are $4!$ ways to permute the columns from a first row of $(1\,2\,3\,4)$, and three ways to permute the other three rows, then 4 squares that can be made, there are $4!3!4 = 4!^2$ total Latin squares.

4. The graph has 4 triangles, each corresponding to one Latin square above. The derangements $(2\,1\,4\,3), (3\,4\,1\,2)$ and $(4\,3\,2\,1)$ each have degree 4, since they appear in two of the squares above, while the other six derangements have degree 2.

5. There are $n!$ permutations of $[n]$ for each row, and we choose one for each row, so an upper bound is $n!^n$.

6. The base case has 1 row remaining, and there is only one way to fill it in. Suppose there are at least $(k-1)!$ ways to fill in a row when each set S_i has at least $k-1$ elements. To fill in the next row, we can choose the last element from any of the k elements in S_n. We then remove this element from every set S_i, leaving them with either k or $k-1$ elements. By the inductive hypothesis, there are at least $(k-1)!$ ways to choose elements from the remaining sets to fill in the row.

7. By the previous argument, there are $n!$ ways to choose the first row, then at least $(n-k)!$ ways for row $k+1$.

8. We can show by induction that

$$\log(1!2!\cdots n!) \geq \frac{1}{2}n^2 \log n,$$

   and that

$$\log(n!^n) = O(n^2 \log n).$$

   Thus the lower and upper bounds on the logarithm of the number of Latin squares are the same order of magnitude.

   This does mean that the upper and lower bounds that we gave on the number of Latin squares are exponentially different. Some improvements can be made with stronger techniques. It is an open problem to improve current estimates.