Tutorial 5
Counting Colourings

MAT 344
Summer 2019
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Learning Objectives

> In this tutorial you will be determining the number of proper colourings for various graphs.
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> These problems relate to the following course learning objectives: *Analyze a counting problem by proving an exact or approximate enumeration, or a method to compute one efficiently*, and *describe solutions to iterated processes by relating recurrences to induction and combinatorial identities.*

Colouring Standard Graphs

1. Suppose you have k colours available to properly colour the vertices of a graph. You do not need to use every colour. Find the number of ways to colour:
   (a) I_1, I_2, I_3, the independent graphs with no edges;
   (b) K_3, K_4, K_5, the complete graphs;
   (c) P_2, P_3, P_4, the path graphs on n vertices.
2. For each graph above, expand your answer as a polynomial in k. What do you notice about the degree and coefficients? What do you notice about the integer roots?

A Recursive Process

For a graph G with an edge xy, we can construct two new graphs:

- $G - \{xy\}$ by deleting the edge, leaving the vertices alone;
- $G/\{xy\}$ by contracting the edge, combining x and y into one vertex, and removing any multiple edges or loops.

For example, deleting any edge in the cyclic graph C_4 gives P_4, and contracting any edge gives C_3.

3. Show that the number of colourings of $G - \{xy\}$ is (the number for G) plus (the number for $G/\{xy\}$).
4. Use this recurrence to compute the number of ways to colour C_4 and C_5.
5. Compare your conjectures from Q2 to these polynomials. Use the recurrence relation or your knowledge of colourings to prove any of them.
6. A graph G has 5 vertices, $\{1, 2, 3, 4, 5\}$ and 5 edges $\{12, 23, 34, 41, 45\}$. Use the deletion and contraction recurrence to compute the number of colourings with k colours. Which edge is easiest to use?

1. (a) These are the same as strings with no restrictions, so we have k, k^2, k^3, and in general k^n ways to colour I_n.
   (b) These are strings with no repeated colours allowed, so there are $P(k,3)$, $P(k,4)$, and $P(k,5)$ ways. Note that if $k < n$, then $P(k,n) = 0$.
   (c) These are strings where no two adjacent colours can be the same. There are k choices for the first vertex (either endpoint), and $k-1$ for each vertex after that. So we have $k(k-1)^{n-1}$ ways to colour P_n.

2. Some things you might notice: the degree of each polynomial is the number of vertices; the leading coefficient is always 1; the coefficients alternate in sign; the negative of the coefficient of k^{n-1} is the number of edges; there is no constant term, they all have 0 as a root; the least positive integer that is not a root is $\chi(G)$;

3. Partition colourings of $G - \{xy\}$ into two disjoint sets: those where x and y get the same colour, and those where they don't. If they get the same colour, then we have a colouring of $G/\{xy\}$. If they don't, it's a colouring of G. Every colouring of either graph on the right gives a colouring of the correct type for $G - \{xy\}$.

4. For C_4, we can use the colourings of P_4 and C_3 to get $k(k-1)(k^2-3k+3)$. For C_5 we use P_5 and C_4 to get

$$k(k-1)(k^3-4k^2+6k-4) = k(k-1)(k-2)(k^2-2k+2).$$

5. We can prove the degree is n and the leading coefficient is 1 by using induction on the number of edges. Deletion and contraction both reduce the number of edges by at least 1, and deletion leaves the number of vertices the same. The base case is I_n, the graph with no edges. After we have these two facts, induction also allows us to prove that the absolute value of the coefficient of k^{n-1} is the number of edges, since deletion removes one edge and keeps the same degree, while contraction reduces the degree by 1. The polynomial from contraction has leading coefficient 1, which makes up for the removed edge in the deletion polynomial.

   Proving the conjectures about the roots is an argument about colouring: $\chi(G)$ is the minimum value where a colouring is possible, so lower non-negative coefficients must have 0 possible colourings, meaning they are roots of the polynomial. Any graph with at least one vertex requires at least one colour, so 0 is always a root.

6. Deleting the edge 45 gives C_4 and one disjoint vertex, so this has polynomial

$$k^2(k-1)(k^2-3k+3),$$

   since the components can be coloured separately, and the number of colourings multiply. Contracting the edge gives C_4, hence the polynomial for G is

$$k(k-1)^2(k^2-3k+3).$$

   We can check that there are 2 colourings with $k = 2$ colours, which is true for any connected bipartite graph.