Tutorial 4

Recursion to Complexity

Learning Objectives

> In this tutorial you will be determining the growth rates of functions given by recurrences. A standard method for computing values of a function is to observe a pattern in low values, determine a recurrence, and prove it by induction. Once we know a recurrence, we can exactly compute $f(n)$ by computing all previous values of $f(k)$ for $k \leq n$, but we often only care about an estimate of $f(n)$, which does not require us to do all of that computation.
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> These problems relate to the following course learning objectives: *Describe solutions to iterated processes by relating recurrences to induction and combinatorial identities*, and *identify when an exact solution is intractable, and use estimates to describe its approximate size.*

Matching recurrences to functions

Match each recurrence relation to the function that satisfies it.

a. $f(n) = 2f(n-1) - n,\ f(0) = 2$	______	$f(n) = \lfloor \log_2 n \rfloor + 1$
b. $f(n) = 2f(n-1) - n,\ f(0) = 3$	______	$f(n) = 2^{\lfloor \log_2 n \rfloor}$
c. $f(n) = f(\lfloor n/2 \rfloor) + 1,\ f(1) = 1$	______	$f(n) = n + 2$
d. $f(n) = f(\lfloor n/2 \rfloor),\ f(1) = 1$	______	$f(n) = 1$
e. $f(n) = 2f(\lfloor n/2 \rfloor),\ f(1) = 1$	______	$f(n) = 2^n + n + 2$

Asymptotic growth rates

For each function described below, give the asymptotic growth rate $O(g(n))$ in terms of a constant, logarithmic, polynomial, or exponential function $g(n)$, and explain why. You can assume that the functions are all strictly increasing to avoid solutions like $f(n) = c$ for all n.

1. $f(n+1) = 3f(n) - 2n + 3$.
2. $f(n+1) = 2f\left(\lfloor \frac{n}{2} \rfloor\right) + 1$.
3. $f(n+1) = 2f(n) - f(n-1)$.
4. $f(n+1) = 2f(n) + f(n-1)$.
5. $f(n) = \frac{\|A^n(\vec{v})\|}{\|\vec{v}\|}$, where A is a 3×3 matrix with real eigenvalues $0 < \lambda_3 < \lambda_2 < \lambda_1$ and $\vec{v}$ is a randomly chosen vector in $\mathbb{R}^3$. (Recall: a 3×3 matrix with 3 real eigenvalues is diagonalizable, and so $\mathbb{R}^3$ has a basis of eigenvectors).
6. $s(n+1) = 3s(n) - 3s(n-1) + s(n-2)$

Investigating homogeneous linear recurrences

7. Show that $f(n) = 2^n$ and $f(n) = 5^n$ are both solutions to the recurrence $f(n) = 7f(n-1) - 10f(n-2)$.

8. Show that $g(n) = A2^n + B5^n$ is also a solution for any A and B.

9. Show that if $f(n) = c^n$ is a solution for some c, then $c = 2$ or $c = 5$.

10. Determine all solutions of the form c^n for $f(n) = -f(n-1) + 6f(n-2)$.

Matching solutions are (c), (e), (a), (d), (b), by checking values up to $f(3)$.

1. The function is approximately tripling in value at each step, so $f(n) = O(3^n)$. When $f(0) = 1$, we have $f(n) = 3^n - n$. Other initial values will give different functions, all of the form $A3^n - n$.

2. This function grows linearly with n, since $f(n)$ is approximately double $f(n/2)$, so is $O(n)$.

3. All functions satisfying this recurrence are linear, $f(n) = An + B$, so $f(n) = O(n)$, since it cannot be constant.

4. This slight change makes the function grow exponentially. It is specifically $O((1+\sqrt{2})^n)$, but $O(3^n)$ also works and is easy to prove by strong induction.

5. By repeated applying the linear transformation, we get a vector close to an eigenvector for λ_1. Hence, $f(n) = O(\lambda_1^n)$.

6. This recurrence is ambiguous, even with the restriction that $s(n)$ is strictly increasing. $s(n)$ can be any function of the form $An^2 + Bn + C$, with A and B not both zero, and with positive leading coefficient. The notation is meant to recall the sequence of squares from PS 3, where $A = 1, B = C = 0$. Any such equation can be shown to work for some initial values by strong induction, so we have $s(n) = O(n^2)$ if $A \neq 0$, or $s(n) = O(n)$ if $A = 0$.

7. By induction, suppose $f(n-1) = 2^{n-1}$ and $f(n-2) = 2^{n-2}$. Then

   $$7f(n-1) - 10f(n-2) = 7 \cdot 2^{n-1} - 10 \cdot 2^{n-2} = 7 \cdot 2^{n-1} - 5 \cdot 2^{n-1} = 2 \cdot 2^{n-1} = 2^n.$$

   Similarly for 5^n.

8. Separate into two parts and factor out A and B for an inductive proof.

9. We use contradiction and inequalities. If $c > 5$ or $c < 2$, then $7c - 10 < c^2$ (consider the graph of the quadratic $x^2 - 7x + 10$), hence $7f(n-1) - 10f(n-2) < f(n)$. Similarly, if $2 < c < 5$, then $7c - 10 > c^2$.

10. Solutions are of the form 3^n or $(-2)^n$.