Tutorial 10

Conditional Probability and the Monty Hall Problem

Learning Objectives

> In this tutorial you will investigate a famous puzzle using conditional probability
>
> These problems relate to the following course learning objective: *Select and justify appropriate tools to analyze a counting problem.*

1 A chance to win a car!

You are a participant in a game show and you're given the choice of three doors: Behind one door is a car; behind the others, goats.

1. You have no information on where the car is, so you have to guess. What is the probability of you winning the car?
2. If you play the game 20 times, how many times do you expect to win?
3. How close to the expectation do you expect to be?
4. Designate one member of your group as the host and another one as the player. Play the game 20 times. Compare the results to what you computed.

We change the rules of the game a bit. Now after the player selects a door, the host, who knows what is behind the doors opens a door revealing a goat (if the host has a choice in which door to open, they do so randomly). Then the player has the opportunity to switch their selection to the unrevealed door.

For example:

- If the car is behind door 1, and the player guesses door 1 first, the host opens door 3, revealing a goat. The player sticks with their choice of door 1 and wins.
- If the car is behind door 2, and the player guesses door 1 first, the host opens door 3, revealing a goat. The player switches to door 2 and wins.

5. Someone argues: "Once the host opens a door, there are two unopened doors and one car, so there is a 50/50 chance of the car being behind either door. Therefore it does not matter if you switch or stick with your choice". Does this argument sound reasonable to you? Discuss in your group.
6. Play the updated game 20 times. Have the non-participating members count the number of times switching wins, and the number of times sticking with the original choice wins. What do you notice? Compare your results with the other groups.

2 Using Conditional Probability

For $i = 1, 2, 3$, Let Ci be the event that the car is behind door i, let Xi be the event that the player's first guess is door i, and let Hi be the event that the host opens door i. Compute the following probabilities, and explain in words what the events mean:

7. $P(Ci)$, $P(Xi)$, $P(Ci \cap Xi)$. Are Ci and Xi independent?
8. $P(H3|C1 \cap X1), P(H3|C2 \cap X1), P(H3|C3 \cap X1)$,
9. $P(H3 \cap X1 \cap C1), P(H3 \cap X1 \cap C2), P(H3 \cap X1 \cap C3)$,
10. $P(H3 \cap X1)$,
11. $P(C1|H3 \cap X1)$, $P(C2|H3 \cap X1)$, $P(C3|H3 \cap X1)$.
12. Assume that you, as the player, select door 1 first, and the host reveals a goat behind door 3. You can decide to switch to door 2 or stick to door 1. Which choice maximizes your chances of winning the car?
13. Discuss your findings within your group. Can you find an explanation not using conditional probability?

1. $\frac{1}{3}$.

2. Let X be a random variable counting the number of cars won in the series of 20 tries. Then $E(X) = \frac{20}{3}$.

3. Let Y be the random variable that is counting the number of cars won in a single play of the game. We compute the variance for Y

$$\begin{aligned} \text{var}(Y) &= E\left(E(Y - E(Y))^2\right) \\ &= \left(\frac{1}{3}\right)\left(\frac{4}{9}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{9}\right) \\ &= \frac{2}{9} \end{aligned}$$

using the linearity of variance,

$$\text{var}(X) = 20\text{var}(Y) = \frac{40}{9}.$$

Therefore the standard deviation for 20 plays is $\sqrt{\frac{40}{9}}$.

4. Activity

5. The argument is wrong because it assumes that the host's choice is independent of the player's initial choice and of the location of the car.

6. Activity

7. $P(Ci) = P(Xi) = \frac{1}{3}$, $P(Ci \cap Xi) = \frac{1}{9}$. This means that Ci and Xi are independent.

8. $P(H3|C1 \cap X1) = \frac{1}{2}$, as the host may select either of the doors with goats,$P(H3|C2 \cap X1) = 1$, since the host must select a door with a goat ,$P(H3|C3 \cap X1) = 0$, as the host can't open the door that's the player's initial choice.

9. Using the previous two parts, $P(H3 \cap X1 \cap C1) = \frac{1}{18}$, $P(H3 \cap X1 \cap C2) = \frac{1}{9}$, $P(H3 \cap X1 \cap C3) = 0$.

10. Using the previous part (since $C1 \cup C2 \cup C3$ is the entire probability space), $P(H3 \cap X1) = \frac{1}{18} + \frac{1}{9} = \frac{1}{6}$.

11. $P(C2|H3 \cap X1) = \frac{P(C2 \cap H3 \cap X1}{P(H3 \cap X1)} = \frac{1/9}{1/6} = \frac{2}{3}$, $P(C1|H3 \cap X1) = \frac{P(C1 \cap H3 \cap X1}{P(H3 \cap X1)} = \frac{1/18}{1/6} = \frac{1}{3}$. $P(C3|H3 \cap X1) = 0$ as the host can't reveal the car.

12. Switching wins if the car is behind door 2, and $P(C2|H3 \cap X1) = \frac{2}{3}$. Sticking with door 1 wins if the car is behind door 1, and $P(C1|H3 \cap X1) = \frac{1}{3}$. So we should switch.

13. One explanation is: Switching wins if we selected a door with a goat first, since then the host will reveal the other goat and we switch to the car. Sticking with our choice wins if we select the door with the car first. Since we have a $\frac{2}{3}$ probability of selecting a door with a goat first, we should switch.