MAT344 H1S - Combinatorics
Midterm Test — March 6, 2019
Time: 110 minutes

Please complete this cover page with ALL CAPITAL LETTERS.

Last name .....

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Student number .....

UTOR ID .....

Instructions: (READ CAREFULLY!)

- Do not write or draw anything on the QR code on the top corner of any page.
- This exam booklet contains 11 pages including this one. It consists of 8 questions.
- In order to get any points, **you need your final answer to be correct and you need a justification,** unless otherwise indicated.
- If you need scratch paper, use the back of the pages. **We will only read and grade what you write on the front of each page.**
- **If you need extra space for a question, you may use Pages 10 and 11 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- No calculators, cellphones, notes, or any aids are allowed.
  If you have any, place them underneath your seat.
- Do not turn over this page until the invigilators instruct you to do so. Good luck!

1. (12 points) Match each description with the number that counts it. The same number may be used more than once. Each letter may only be used once.

A. The number of binary strings of length 6 with 5 zeroes.

B. The number of ways to put 5 identical items into 6 distinguished boxes.

C. The number of ways to arrange 6 distinguished boxes in a line.

D. The number of length 3 ternary strings with no repetition.

E. The number of lattice paths from $(0, 0)$ to $(5, 5)$, only going up or right.

F. The number of possible ways to answer Q1 by using each letter once.

6^5 _____

5^6 _____

$\binom{5+6-1}{5}$ _____

$\binom{6}{5}$ _____

$P(6, 5)$ _____

2. (8 points) How many integer solutions exist to the equation

$$a + b + c + d + e = 127$$

where a, b, c, d, e all have to be at least 2?

3. (10 points) Provide a bijective (combinatorial) argument for the following identity:

$$3^n - 2^n = \sum_{i=0}^{n-1} 2^i 3^{n-1-i}$$

4. (10 points) An **eulerian walk** is a sequence of vertices in a graph such that every edge is traversed exactly once. It differs from an eulerian circuit in that the starting and ending vertex don't have to be the same.

   Prove that if a graph is connected and has at most two vertices of odd degree, then it has an eulerian walk.

5. (10 points) A *genetic sequence* is a string over the 4-letter alphabet $\{A, T, C, G\}$.

   (a) (4 points) Show that the number of genetic sequences of length n with no two consecutive repeated letters (AA, CC, GG, or TT) is $4 \cdot 3^{n-1}$.

   (b) (4 points) Give a recurrence relation and initial conditions for $b(n)$, the number of length n genetic sequences with no *three* consecutive repeated letters.

   (c) (2 points) Is $b(n) = O(3^n)$?

6. (10 points) A certain sequence of numbers t_n follows the pattern $t_{n+1} = 3t_n - 3t_{n-1} + t_{n-2}$ when $n > 2$, and has $t_1 = 0$, $t_2 = 1$ and $t_3 = 3$. Show that $t_n = \binom{n}{2}$.

7. (10 points) At most how many elementary operations would an algorithm have to perform in order to tell whether an n-vertex graph has a subgraph that is isomorphic to K_k (i.e. there are k vertices that are all connected to each other)? Elementary operations include determining whether there is an edge present between two vertices.

8. (10 points) How many ways are there to choose a 5-letter "words" from the 26-letter English alphabet with repeated letters allowed, but words that are anagrams are considered the same? Anagrams are words that have the same collection of letters (but in a different order). For example, TREES is an anagram of RESET, and AAABB is an anagram of BABAA. The words need not be English language words.

Use this page if you run out of space for one or more problems. If you do so, clearly indicate in the corresponding problem page that you will continue here.

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