MAT 344 Concepts and problems (for the final)

How to use this document

This course has six learning outcomes that students should demonstrate:

- Select and justify appropriate tools (recurrences, generating functions, graph algorithms) to analyze a counting problem.
- Analyze a counting problem by proving an exact or approximate enumeration, or a method to compute one efficiently.
- Describe solutions to iterated processes by relating recurrences to induction or combinatorial identities.
- Identify when an exact solution is intractable, and use estimates to describe its approximate size.
- Prove combinatorial identities by counting a set of objects in two ways.
- Construct counting problems which show the usefulness or limitations of combinatorial tools.

For each of these objectives, you have been given problem set and tutorial questions to practice them.

The major topics we covered after the midterm are:

- Planar graphs
- Inclusion-Exclusion
- Generating Functions
- Recurrence Equations
- Probability
- Graph algorithms (minimal spanning trees, Dijkstra's algorithm)
- Networks and the max flow/min cut theorem

Out of these, Inclusion-Exclusion, Generating Functions and Solving Recurrence equations were our main technical tools and the rest of the topics are applications. Make sure you have a good understanding of the technical tools, as they will surely show up in some form.

Look back at your problem sets and tutorials and decide how you would rate your abilities for each topic. Make a list of topics that you want to focus on improving. For each topic in your list, construct a problem that would demonstrate one of the outcomes. Now you have a personalized midterm review problem set that you can practice solving. You can also go over the problems in the midterm review sheet.

To practice the graph algorithm problems that we covered in the last few weeks, our textbook has some great exercises. For example, try Exercises 2,4,7 and 11 from Chapter 12, and Exercises 2,3,5, and 9 from Chapter 13.

These review problems were written by your peers. For each problem, select and justify appropriate tools to analyze it. Depending on what is appropriate, proceed to analyze, describe solution sizes, or prove identities. Feel free to post your solutions, or ask for hints, on the discussion forum.

Various problems

1. Find two nonisomorphic trees with the same degree sequence.
2. There are five bus stations in the GTA area. Each of them has routes to all the other stations. Is it possible that no routes intersect (except at a station)?
3. There are 50 people in a room, 25 people in white T-shirts, 30 people in black pants, 10 people in white T-shirts and black pants. What is the number of people who are not wearing a white T-shirt or black pants?
4. Find the number of positive integers between 1 and 1000 that are not divisible by 13 or 21.
5. There are 100 people waiting for the concert. The administrator decides to give cokes and sprites to them. He gives a can of coke each to every 3rd person in line, he then gives a can of sprite to every 5th person in line. How many people did not receive anything?
6. In how many ways can 18 (identical) cars be parked in 4 garages such that each garage has no more than 7 cars.
7. What is the number of different pattern locks using a 3 by 3 keypad that use each number only once?
8. How many permutations of n contain no i followed by an $i+1$?
9. Find the number of ways of choosing n different socks colored red, yellow and green. We have to choose at most 3 red socks, an odd number of yellow socks and an even number of green socks.
10. Find the number of strings of length n formed from the set $\{a, b, c, d\}$ if there must be at most 2 a's and the number of c's must be even.
11. Find the generating function for the number of solutions to $x_1 + x_2 + x_3 = n$ where x_1 is a multiple of 4, x_2 is any number and x_3 is even.
12. Find the number of binary strings of length n with an even number of 0s.
13. In D&D 5th edition, a longsword deals 1d8 damage which means it's an 8-sided die from 1 to 8. In how many ways can you deal 50 damage exactly using a longsword? (to answer this, it is sufficient to just find the generating function, and not the coefficient. Or, you can use sagemath to find the coefficient)
14. Use generating functions to find the general solution of the recurrence equation
$$g_{n+2} = 2g_{n+1} - g_n.$$
15. In Minecraft, players can interact with different creatures in the world. One of them is sheep whose wool can be sheared for other uses. One player is making a sheep pen and is selling the wool to other players. They can dye the sheep's wool 16 different colors. The player receives an order for n sheep's worth of wool where an odd number of sheep are blue, an even number of sheep are red, and any number of sheep are yellow. In how many ways can the player dye the sheep? (to answer this, it is sufficient to just find the generating function, and not the coefficient. Or, you can use sagemath to find the coefficient)

16. A group of 20 go to lunch. Everyone pitches in \$1 or \$2 except for George and Larry, who will pay either \$4, \$5, or \$6. How many ways are there to split the bill of \$50?

17. How many ways are there of spending n dollars on fruit if bananas cost \$2 each, apples are \$3 each and oranges are \$1 each but the store has only 2 oranges left?

18. Find the general solution to the recurrence equation
$$(A+2)(A-5)(A-1)f(n) = 5^n.$$

19. Solve the recurrence $x_n = 11x_{n-1} - 30x_{n-2}$ with initial conditions $x_0 = 4, x_1 = 23$.

20. Find the number of ways of tiling a $2 \times n$ rectangle with blocks of size 2×1 and 2×2.

21. Find all solutions to the advancement operator equation
$$(A+5)(A-1)^3 f(n) = 0.$$

22. Find the closed form of the recurrence
$$g_n = 3g_{n-1} - 2g_{n-2},$$
where $g(1) = 1, g(2) = 5$.

23. Solve the recurrence
$$g(n) = \frac{3}{2}g(n-1) - \frac{1}{2}g(n-3),$$
where $g(0) = 1, g(1) = 3, g(2) = 9$.

24. Find a closed form solution for the number of ternary sequences of length n that do not contain a 01.

25. There is a staircase to climb, at each step you can take 1 stair or 2 stairs. Find a recurrence relation and a closed form solution for the number of ways to take n stairs.

26. Suppose you design a role-play video game. For each level completed, the player will have ability points equal to the ability points at the start of the current level plus 3 times the ability points at the start of the previous level. The initial ability points are 1 on level 1 and 2 on level 2. You want to know how many ability points a player will have at the time they reach level 100, but sadly you left your laptop in your office so you only have pencil and paper.

27. Find a closed form solution for the number of ternary sequences that do not contain adjacent 2's.

28. You have three different candies and ten of each. One candy is red the others blue and green. What is the probability of you randomly handing out 5 green, 2 blue and 3 red candies to a friend?

29. Suppose we have a deck of cards shuffled. We pick 5 cards. What is the probability that they are all diamonds?

30. There are 15 marbles in a bag, 10 red and 5 blue. Four marbles are selected at random, what is the probability that all of them are blue?

31. A group of 9 people randomly sit down around a round table. In the group there is a brother and a sister. What is the probability that they sit together?

32. What is the probability that the sum of the faces of two fair dice rolls exceed 6?

33. In Texas Hold'em Poker, the players each receive 2 cards from a standard deck of 52 cards and then 5 cards are dealt on the table by the dealer (the dealer first deals 3 cards, then the fourth one next and then the last one). The hand that a player has is determined by the best Poker hand that can be formed with any combination of the 2 cards the player has and the 5 cards on the table. A flush is any hand that has all cards the same suit. Balazs is playing poker with the TAs, what is the probability that he gets a flush on the last card on the table?

34. Suppose you choose integers x and y randomly between 0 and 20 (equal likelihood), what is the probability that $x + y = 10$?