MAT 344 Concepts and problems (up to midterm)

How to use this document

This course has six learning outcomes that students should demonstrate:

- Select and justify appropriate tools (induction, graphs, recurrences, complexity theory) to analyze a counting problem.
- Analyze a counting problem by proving an exact or approximate enumeration, or a method to compute one efficiently.
- Describe solutions to iterated processes by relating recurrences to induction or combinatorial identities.
- Identify when an exact solution is intractable, and use estimates to describe its approximate size.
- Prove combinatorial identities by counting a set of objects in two ways.
- Construct counting problems which show the usefulness or limitations of combinatorial tools.

Note that we have removed references to future material. For each of these objectives, you have been given problem set and tutorial questions to practice them. **Before reading further**, look back at your problem sets and tutorials and decide how you would rate your abilities for each outcome. Make a list of topics and outcomes that you want to focus on improving. For each topic in your list, construct a problem that would demonstrate one of the outcomes. Now you have a personalized midterm review problem set that you can practice solving.

The problems in the rest of the document were written by your peers (including students who have taken MAT344 in previous semesters). For each problem, select and justify appropriate tools to analyze it. Depending on what is appropriate, proceed to analyze, describe solution sizes, or prove identities.

Various problems

1. How many U of T student numbers can be assigned?
2. There are 400 students in MAT344. Let's suppose BA1130 has 10 rows of seats and for each row there are 40 seats. On May 14th, 4 students are absent due to illness. How many possible ways are there for students to sit in the lecture hall?
3. For which n does the graph K_n have an Euler circuit?
4. There are 100 exams to be graded by 5 TAs. The minimum amount a TA can grade is 1 paper. How many different allocations are there?

5. A robocaller places 100 automated calls per minute if they target a single area code (3 digits of a standard 10 digit phone number). What percentage of possible phone numbers have they called in a single day? How many days before it finishes calling everyone in an area code?

6. If 20 students are taking a class photo and two of the students are best friends and must stand next to each other, how many ways are there of ordering the students for the photo?

7. How many ternary strings s of length n satisfy $s_i \neq s_{i+1}$ for each $i < n$?

8. Show that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

9. Give a combinatorial proof of the identity

$$\sum_{k=0}^{n} \left(\binom{n}{k}\right)^2 = \binom{2n}{n}.$$

10. Prove by induction or a combinatorial argument that

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$$

11. What is the sum of all of the terms in Pascal's triangle, up to row n?

12. Approximate the number of binary strings of length $2n$ that don't contain n consecutive zeroes.

13. There are 31 types of pizza listed on pizziaolo.com. How many ways can I order if I order pizza twice a day over a week with no repeats?

14. If the Toronto Raptors play the Golden State Warriors in this year's NBA Finals, and the Raptors win the Finals, how many ways are there if the Raptors never lost two games in a row? The winner is best 4 of 7 games.

15. How long would it take to find the closest pair of points if given a set as (x, y) coordinates?

16. A milktea shop sells different types of milktea. There are 24 kinds of flavours (including original) and 8 kinds of tapioca. Every cup of milktea can have one flavour and $0-2$ kinds of tapioca. How many kids of milktea can they make?

17. Arrange 15 balls of equal size in an equilateral triangle with 5 rows. How many different colours are needed to paint them so that adjacent balls are different colours?

18. Show that $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.

19. Count the number of lattice paths from $(0, 0)$ to (n, n) of length $3n$. Backward movement is allowed.

20. Approximate the number of subsets of size k from n elements if: n is very large compared to k, or if $k = O(n)$.

21. Can a knight in chess travel around the board and go back to its starting position without moving to the same square twice?

22. There are 5 identical balls in a box. How many ways are there to choose 3 of them?

23. Show that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

24. Determine how to find an optimal wedding seating plan using the minimum number of tables where guests who are incompatible with each other cannot sit together.

25. At least how many students will end up with the same final mark in this course MAT 344 with 394 students?

26. Consider $f_k(x,y) = \begin{cases} 1 & \text{if } x < 2 \text{ or } y < k \\ k f_k(x, \lfloor \frac{y}{k} \rfloor) + f_k(\lfloor \frac{x}{2} \rfloor, y) & \text{otherwise} \end{cases}$

    What is $f_{123}(123456789, 987654321)$?

27. What's the runtime complexity of inversion of a n by n matrix?

28. If you have 4 nickels, 3 quarters, 7 loonies, and 2 ten dollar bills, how many different total values are possible?

29. What is the maximum number of edges in an eulerian graph with n vertices?