MAT344 Problem Set 8
(due 1pm July 23)

Note: For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.

Part A

In these problems, you will *analyze a counting problem by proving an exact or approximate enumeration*. Two of these problems will be marked.

Problem 1. For each of the infinite sequences below, give its generating function in closed form (i.e., not as an infinite sum). For example, for the sequence $(1, 1, 1, 1, \ldots)$, the generating function is $\frac{1}{1-x}$.

1. $[2^8, 2^7\binom{8}{1}, 2^6\binom{8}{2}, \ldots, \binom{8}{8}, 0, 0, 0, \ldots]$.
2. $[0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, \ldots]$.

Problem 2. Use a computer algebra system (for example, SageMath following example 8.6 in the textbook, Python, Mathematica, MATLAB) to find the coefficient of x^{500} for the following generating functions:

1. the number of ways of making change in Canada,
$$\frac{1}{(1-x^5)(1-x^{10})(1-x^{25})(1-x^{100})(1-x^{200})}.$$
2. the number of ways of buying chicken McNuggets in Canada,
$$\frac{1}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{20})}.$$

Problem 3. Find the generating function for the number of non-negative integer solutions to
$$3x + 2y + 7z = n$$

Problem 4. Find an exponential generating function and the coefficient of $\frac{x^n}{n!}$ for the number of permutations with repetition of length n of the set $\{a, b, c\}$, in which there are an odd number of as, any number of bs, and an even number of cs.

Part B

In these problems, you will *analyze a counting problem by proving an exact or approximate enumeration* and *construct counting problems which show the usefulness or limitations of combinatorial tools.*. Two of these problems will be marked.

Problem 5. For the recursively-defined sequence $c_0 = 2, c_1 = 0$, $c_n = c_{n-1} + 2c_{n-2}$ for $n \geq 2$, use the method of generating functions to find an explicit formula for the nth term of the sequence.

Problem 6.

(a) Find an ordinary or exponential generating function for the number of ways of painting some number of identical balls red, yellow and blue, if at most 2 can be painted red, an even number must be painted yellow, and any number can be painted blue.

(b) Use the generating function you found to answer the question for n balls.

Problem 7.

(a) Find an ordinary or exponential generating function for the number of ways of painting some number of different hotel rooms red, yellow and blue, if at most 2 can be painted red, an even number must be painted yellow, and any number can be painted blue.

(b) Use the generating function you found to answer the question for n rooms.

Part C

In this problem, you will *construct counting problems which show the usefulness or limitations of combinatorial tools.* This problem will be marked for completeness only.

Problem 8. Construct a counting problem that can be solved using generating functions, but is difficult to solve using counting techniques you've learned earlier.