MAT344 Problem Set 7
(due 1pm July 16)

Note: For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.

Part A

In these problems you will *analyze a counting problem by proving an exact or approximate enumeration*. Two of these problems will be marked.

Problem 1. How many surjections are there from [7] to [4]? (for this question, give the answer as a number, not a formula)

Problem 2. How many derangements are there of the set [6]? (for this question, give the answer as a number, not a formula)

Problem 3. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 32$$

with $0 \leq x_i \leq 10$ for $i = 1, 2, 3, 4$?

Problem 4. How many numbers from 1 to 5000 are divisible by either 3 or 17?

Part B

In these problems, you will *select and justify appropriate tools to analyze a counting problem*. Two of these problems will be marked.

Problem 5. The **Euler totient function** is a function $\phi(n)$ that is defined as follows: for a positive integer $n \geq 2$, let

$$\phi(n) = |\{m \in \mathbb{Z} | 1 \leq m \leq n, gcd(m, n) = 1\}|$$

(that is, $\phi(n)$ is the number of positive integers less than or equal to n relatively prime to n).

(a) Find $\phi(p)$ for p a prime number.

(b) Find $\phi(p^k q^l)$ for p and q distinct primes and k, l positive integers.

(c) Use inclusion-exclusion to find $\phi(n)$ for $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ with the p_i distinct primes and k_i positive integers.

Problem 6. The principle of inclusion-exclusion is not the only approach available for counting derangements. We know that $d_1 = 0$ and $d_2 = 1$. Using this initial information, it is possible to give a recursive form for d_n. Give a combinatorial argument to prove that the number of derangements satisfies the recursive formula

$$d_n = (n-1)(d_{n-1} + d_{n-2})$$

for $n \geq 2$.

Problem 7. A small merry-go-round has 8 seats occupied by 8 children. In how many ways can the children change places in a way that no child sits behind the same child as on the first ride? The seats do not matter, only the relative positions of the children.

Part C

In this problem, you will *construct counting problems which show the usefulness or limitations of combinatorial tools.* This problem will be marked for completeness only.

Problem 8. Give an example of a problem that is difficult to enumerate directly but can be counted using inclusion-exclusion.