MAT344 Problem Set 5
(due 1pm June 11)

Note: For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.

Part A

In these problems, you will *select and justify appropriate tools to analyze a counting problem* and *analyze a counting problem by proving an exact or approximate enumeration.* Two of these problems will be marked.

Problem 1. A pharmaceutical manufacturer is building a new warehouse to store its supply of 10 chemicals it uses in production. However, some of the chemicals cannot be stored in the same room due to undesirable reactions that will occur. The matrix below has a 1 in position (i, j) if and only if chemical i and chemical j cannot be stored in the same room. Model this problem using graph theory and determine the smallest number of rooms into which they can divide their warehouse so that they can safely store all 10 chemicals in the warehouse.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problems 2 and 3 refer to the graph in Figure 1. Each vertex is labelled by a letter (i.e. $a, b, c, \ldots$), and each edge is labelled by an integer (i.e. $1, 2, \ldots$).

Problem 2. Is the graph in Figure 1 Eulerian? If it is, find an Eulerian circuit using the edge labelling. If it is not, explain why it is not.

Problem 3. Is the graph in Figure 1 Hamiltonian? If it is, find an Hamiltonian circuit using the edge labelling. If it is not, explain why it is not.

Problem 4. There are 11 non-isomorphic graphs on 4 vertices. Draw all 11, and under each one indicate: is it connected? Is it a forest? Is it a tree? **Hint:** One has 0 edges, one has 1 edge, two have 2 edges, three have 3 edges, two have 4 edges, one has 5 edges and one has 6 edges.

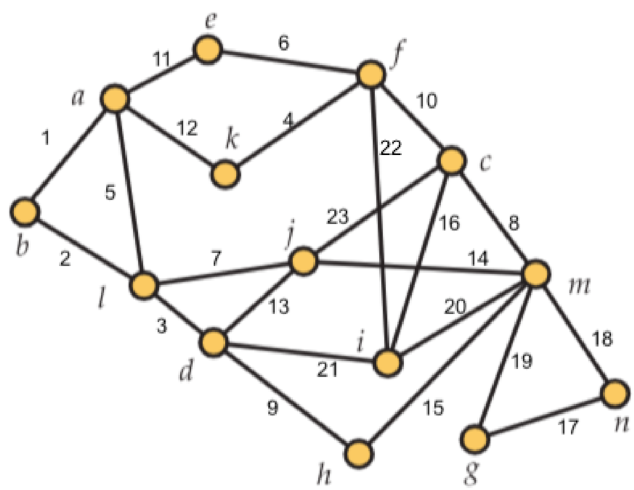

Figure 1: A graph

Part B

In these problems, you will *analyze a counting problem by proving an exact or approximate enumeration.* Two of these problems will be marked.

Problem 5. Use graphs to give a combinatorial proof that

$$\sum_{i=1}^{k} \binom{n_i}{2} \leq \binom{n}{2},$$

where $n_1, n_2, \ldots, n_k$ are positive integers with $\sum_{i=1}^{k} n_i = n$ and we define $\binom{1}{2} = 0$. Under what circumstances does equality hold?

Problem 6. Consider the four graphs on Figure 2: Determine which (if any) pairs of graphs

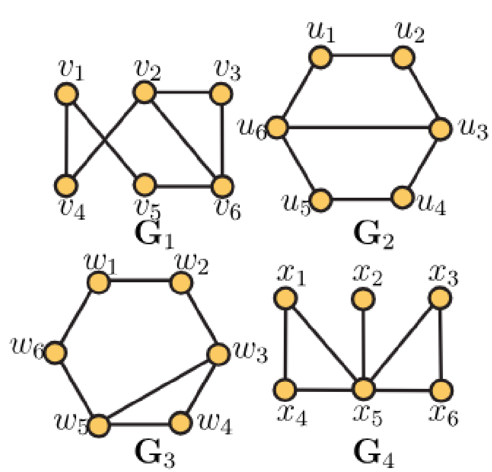

Figure 2: Four graphs

are isomorphic. For pairs that are isomorphic, give an isomorphism between the two graphs. For pairs that are not isomorphic, explain why.

Problem 7. The **complement** $\overline{G}$ of a graph G is the graph with the same vertices as G and xy is an edge of $\overline{G}$ if and only if it is not an edge of G. A graph is **self-complementary** if G is isomorphic to $\overline{G}$. Show that if G is self-complementary then it has $4k$ or $4k+1$ many vertices for some nonnegative integer k. Find a self-complementary graph on 4 vertices and one on 5 vertices

Part C

This question relates to the learning outcomes *construct counting problems which show the usefulness or limitations of combinatorial tools.* It will be marked for completeness only.

Problem 8. Give an example of two nonisomorphic graphs with the same degree sequence.