MAT344 Problem Set 3
(due 1pm May 28)

Note: For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.

Remember: To give a recursive formula you have to relate values of a certain function $f(n)$ defined over the natural numbers to values $f(k)$ of the functions with $k < n$.

Part A

These questions relate to the learning outcomes *select and justify appropriate tools to analyze a counting problem* and *describe solutions to iterated processes by relating recurrences to induction, generating functions, or combinatorial identities.* Two of these questions will be marked.

Problem 1. Give a recursive formula for the number of ways P_n of parenthesizing n factors. For example, here are all the ways of parenthesizing 4 factors:

$$a(b(cd)), a((bc)d), (ab)(cd), ((ab)c)d, (a(bc))d$$

Problem 2. Consider a $1 \times n$ checkerboard. The squares of the checkerboard are to be painted white and gold, but no two consecutive squares may both be painted white. Let $p(n)$ denote the number of ways to paint the checkerboard subject to this rule. Find a recursive formula for $p(n)$ valid for $n \geq 3$.

Problem 3. Let f be a function on the nonnegative integers such that $f(0) = 0$ and $f(n) = n + f(n-1)$. Use mathematical induction to prove that $f(n) = \frac{n(n+1)}{2}$.

Problem 4. There are n hungry lions on an island and a piece of meat. The meat is poisoned, and any lion eating it will fall asleep. A lion who is asleep because of the poison may be eaten by another lion, but the poison stays potent and will still cause that lion to fall asleep (at which point they may be eaten by another lion). The first lion has a choice of either eating the meat or not. They are hungry, but they would prefer to stay hungry over being eaten. Should they eat the meat? **Hint: Think recursively. What does this problem have to do with n?**

Part B

These questions relate to the learning outcomes *prove combinatorial identities by counting a set of objects in two ways*, *select and justify appropriate tools to analyze a counting problem* and *describe solutions to iterated processes by relating recurrences to induction, generating functions, or combinatorial identities.* Two of these questions will be marked.

Problem 5. Find a combinatorial proof of the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_{n+1},$$

where F_n denotes the n-th Fibonacci number. **Hint:** Think about the domino tiling interpretation of F_n discussed in lecture.

Problem 6. Use induction to prove that for all integers $n \geq 1$, the quantity

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9.

Problem 7. A $2 \times n$ checkerboard is to be tiled using three types of tiles. The first tile is a white 1×1 square tile. The second tile is a red 2×2 tile and the third one is a black 2×2 tile. Let $t(n)$ denote the number of tilings of the $2 \times n$ checkerboard using white, red and black tiles.

(a) Find a recursive formula for $t(n)$ and use it to determine $t(7)$.

(b) Let $f(n) = c_1 2^n + c_2(-1)^n$. Determine c_1 and c_2 so that $f(0) = f(1) = 1$.

(c) Prove that $f(n)$ satisfies the same recurrence relation as $t(n)$.

(d) Can we now conclude that $f(n) = t(n)$ for all positive integers n?

Part C

This question relates to the learning outcomes *construct counting problems which show the usefulness or limitations of combinatorial tools.* It will be marked for completeness only.

Problem 8. Give an identity which can be proved by induction or by a combinatorial argument. Which do you prefer?