MAT344 Problem Set 10
(due 1pm Aug. 6)

Note: For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.

Part A

In these problems, you will *analyze a counting problem by proving an exact or approximate enumeration*. Two of these problems will be marked.

Problem 1. Suppose you choose three integers x, y, z randomly (with equal likelihood) between 0 and 50 inclusive, with repeats allowed. What is the probability that

(a) $x + y + z = 25$

(b) x, y, z are all at most 10, given that $x+y+z = 25$. (**Hint:** let $a = 10-x, b = 10-y, c = 10-z$)

Problem 2. Suppose you roll two standard six-sided dice, with sides labelled $1-6$. Find the expected value of the product of the numbers on the two faces.

Problem 3. In a standard deck of cards, there are 4 suits (Clubs, Hearts, Diamonds, and Spades) and 13 ranks of each suit (2 through 10, Jack, Queen, King, Ace). The diamonds and hearts are red, spades and clubs are black. Imagine drawing cards (without replacement) from a shuffled deck, so that any card in the deck is equally likely to be drawn. What is the probability that

(a) If you draw 2 cards, you get both

- an Ace,
- a Jack, Queen or King?

(b) If you draw 5 cards, there are cards from at least three suits in your hand?

Problem 4. If you toss a standard coin 1000 times and let X count the number of heads, how close do you expect X to be to 500?

Part B

In these problems, you will *select and justify appropriate tools to analyze a counting problem*, *analyze a counting problem by proving an exact or approximate enumeration*, and *Identify when an exact solution is intractable, and use estimates to describe its approximate size.* Two of these problems will be marked.

Problem 5. A random graph with vertex set $\{1, 2, \ldots, 10\}$ is constructed using the following method. For each two element subset $\{i, j\}$ from $\{1, 2, \ldots, 10\}$, a fair coin is tossed and the edge $\{i, j\}$ then belongs to the graph when the result is "heads." For each subset $S \subseteq \{1, 2, \ldots, 10\}$ let E_S be the event that S is a complete subgraph in our random graph.

(a) Explain why $P(E_S) = 1/8$ for each 3-element subset S.

(b) Explain why E_S and E_T are independent when $|S \cap T| \leq 1$.

(c) Fix $3 \leq k \leq 10$. What is the expected number of k-subsets $S \subseteq [10]$ such that E_S happens?

Problem 6. Suppose that a test for a particular drug will show positive results with probability 0.95 for users of the drug and show negative results with probability 0.95 for people who do not use the drug. Given that a person tests positive, what is the probability that they are using the drug if 5% of the populations are drug users? (**Hint:** Try to use $P(A \cap B) = P(A|B)P(B)$ many times)

Problem 7. A dice game is played as follows: you pay one dollar to play, then you roll a fair six-sided die. If you roll a six, you win three dollars. Someone claims to have won a thousand dollars playing this game nine thousand times. How unlikely is this? Find an upper bound for the probability that a person playing this game will win at least a thousand dollars.

Part C

In this problem, you will *construct counting problems which show the usefulness or limitations of combinatorial tools.* This problem will be marked for completeness only.

Problem 8. Describe a probability question that can be solved by counting.