**Problem 1.** [6 points] Give a combinatorial proof for the formula, where $x, y \in \mathbb{N}$.

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}$$

**Solution:**

- (2 points) The right hand side $\binom{x+y}{n}$ naturally counts $n$-element subsets of the set $[x+y]$. The left hand side can be interpreted as counting pairs of subsets $(S_1, S_2)$ with $S_1 \subseteq [x], S_2 \subseteq [y], |S_1| + |S_2| = n$. We will set up a bijection between the two sides.

- (2 points) Given an $n$-subset $S \subseteq [x+y]$, the intersection $S \cap [x]$ is a subset of $[x]$. Let $k = |S \cap [x]|$. Clearly $0 \leq k \leq x$. The complementary subset $S \setminus [x]$ is a subset of $\{x+1, x+2, \ldots, x+y\}$ with $n-k$ many elements. We may identify $S \setminus [x]$ with a subset of $[y]$ by subtracting $x$ from each element (subtracting a constant from every element of a set is clearly a bijection). So we defined a map $f$ sending an $n$-subset $S$ of $[x+y]$ to a pair of subsets $(S_1, S_2)$ with $S_1 \subseteq [x], S_2 \subseteq [y], |S_1| + |S_2| = n$.

- (2 points) We now show that $f$ is a bijection. To show that $f$ is surjective, note that given any pair of subsets $(S_1, S_2)$ with the conditions above, we can form the set $S_1 \cup (S_2 + x)$, where $S_2 + x = \{j + x \mid j \in S_2\}$, and $S_1 \cup (S_2 + x) \subseteq [x+y]$ with $|S_1 \cup (S_2 + x)| = n$. We clearly have $f(S_1 \cup (S_2 + x)) = (S_1, S_2)$. To show that $f$ is injective, note that if $f(A) = f(B)$ for $n$-subsets $A, B$ of $[x+y]$, then we must have $A \cap [x] = B \cap [x]$, and also $A \cap ([n] \setminus [x]) = B \cap ([n] \setminus [x])$. Hence
$$A = (A \cap [x]) \cup (A \cap ([n] \setminus [x])) = (B \cap [x]) \cup (B \cap ([n] \setminus [x])) = B$$
so $f$ is injective.

**Solution:**

- (2 points) The right hand side $\binom{x+y}{n}$ naturally counts binary strings of length $x+y$ with exactly $n$ 1s. The left hand side can be interpreted as counting pairs of binary strings $(S_1, S_2)$ of length $x$ and $y$ with a combined number of $n$ 1s. We will set up a bijection between the two sides.

- (2 points) Given a binary string of length $x+y$ with exactly $n$ 1s, we can separate it into two strings, the initial $x$-string $S_{\leq x}$ and the terminal $y$-string $S_{>x}$. Clearly the toal number of 1s is unchanged by this, so this defines a map $f$ from binary strings of length $x+y$ with $n$ 1s to pairs of binary strings with a combined number of $n$ 1s.

- (2 points) We now show $f$ is a bijection. We will do this by defining an inverse map $g$. Given a pair $(S_1, S_2)$ of binary strings of lengths $x$ and $y$ with a combined total of $n$ 1s, let $g(S_1, S_2) = S_1 S_2$ (the concatenation of the two strings). This defines a binary string of length $x+y$ with $n$ 1s. Also, we have
$$g(f(S)) = g(S_{\leq x}, S_{>x}) = S_{\leq x} S_{>x}$$
and
$$f(g(S_1, S_2)) = f(S_1 S_2) = ((S_1 S_2)_{\leq x}, (S_1 S_2)_{>x}) = (S_1, S_2),$$
where the last equality follows from the fact that $|S_1| = x$ and $|S_2| = y = (x+y) - x$. Since $f$ has a two-sided inverse, it must be a bijection.

**Problem 2.** Find a recurrence relation for the number $g(n)$ of ternary strings (that is, $\{0, 1, 2\}$-strings) of length $n$ that do not contain 102 as a substring.

**Solution:**

- (1 point) Any ternary string of length $n$ can be obtained from a ternary string of length $n-1$ by appending a character (a 0,1, or 2) at the end.
- (3 points) Given a ternary string of length $n-1$ that does not contain 102 as a substring, we can get an $n$-string from it by appending a character at the end. The resulting string will not contain the substring 102, unless we append a 2 at the end of an $n-1$ string that ends with 10. There are $g(n-3)$ many such ternary strings that do not contain 102 as a substring.
- (2 points) Therefore, the recurrence relation is

$$g(n) = 3g(n-1) - g(n-3).$$