LAST (Family) NAME: _____

FIRST (Given) NAME: _____

Email address: _____ @mail.utoronto.ca

STUDENT NUMBER: _____

University of Toronto
Faculty of Arts and Sciences

MAT344 PRACTICE EXAM

Duration: 3 hours
Aids Allowed: None

Exam Format and Grading Scheme:

Answers must be written on the examination paper.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90

Students must hand in all examination materials at the end

1. (10 points) Determine the closed form expression for the n'th term of each generating function:

   (a) $\frac{3x}{1-4x} + \frac{1}{1-x}$

   (b) $(3+x)^6$

   (c) $\frac{1}{1+x^3}$

   For each of the following sequences, find a closed form expression for the corresponding generating function (that does not involve an infinite sum). The sequences begin with index 0.

   (c) $[1, 3, 5, 7, 9, \ldots, 2k+1, \ldots]$

   (d) $[1, 4, 9, 16, 25, \ldots, k^2, \ldots]$

2. (10 points) Suppose there are five shows on your Netflix list. Three have 25 minute episodes, the other two have 50 minute episodes. You will watch the episodes of each show in order, but you can change shows after each episode, and watch shows in any order. Suppose you have $25n$ minutes to watch tv between studying. Let a_n be the number of ways you can watch these shows during that time.

   (a) Give a recurrence and initial conditions for a_n.

   (b) Use your recurrence and initial conditions to find a closed formula for a_n depending only on n. (If your solution to part (a) is incorrect, you will still receive marks for analysing it correctly here.)

3. (10 points) Give a bijective proof of the equality

$$3^n - 2^n = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$$

4. (10 points) Simone has 12 mint condition, limited edition collector cards of famous mathematical theorems (each card has a different theorem on it). She wants to display them in 4 distinct display cases. However, she can't decide how to distribute them, so she chooses a uniformly random assignment of cards, out of all the ways of distributing the cards so that each display gets at least one card.

(a) How many different ways can Simone distribute the cards?

(b) What is the probability that every display gets 3 cards?

(c) What is the probability that one display gets 9 cards?

5. (10 points) For the folowing graph, determine:

   (a) if it is Eulerian;

   (b) if it is Hamiltonian;

   (c) if it is planar;

   (d) its chromatic number;

   (e) its maximum clique size.
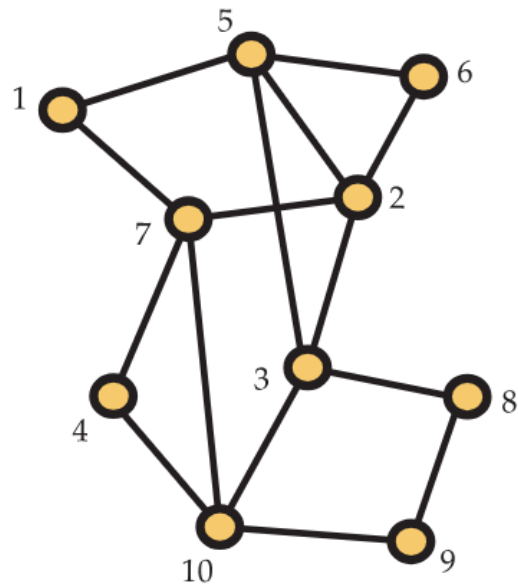
6. (10 points) Show that there is no graph with at least 2 vertices where all the vertices have distinct degrees.

7. (10 points) A set of integers A contains an *arithmetic progression of length* 3 if there exist integers a and $r \neq 0$ such that A contains $\{a, a + r, a + 2r\}$. Suppose Bob is given a set of n integers, and is able to do elementary arithmetic operations (add, multiply, subtract, divide) with numbers in the set. Approximately how many arithmetic operations does Bob need to perform in order to determine if his set contains an arithmetic progression of length 3?

8. (10 points) Suppose we choose integers a, b, c, d from the set of non-negative integer solutions to

$$a + b + c + d = 100$$

uniformly at random. What is the expected value of $a + b$?

9. (10 points) Show that the function $f(n) = 3^n - n2^n$ satisfies the recurrence $f(n) = 7f(n-1) - 16f(n-2) + 12f(n-3)$ with initial conditions $f(0) = f(1) = f(2) = 1$.