

LAST (Family) NAME: _____

FIRST (Given) NAME: _____

Email address: _____@mail.utoronto.ca

STUDENT NUMBER: _____

University of Toronto
Faculty of Arts and Sciences

MAT344 PRACTICE EXAM

Duration: 3 hours
Aids Allowed: None

Exam Reminders:

- Fill out your name and student number at the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. If it is left in your pocket, it may be an academic offence.
- When you are done with your exam, raise your hand for someone to come and collect your exam. Do not collect your bag and jacket before your exam is handed in.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special Instructions:

- If you need scratch paper, use the back of the pages. **We will only read and grade what you write on the front of each page.**
- **If you need extra space for a question, you may use Pages 10 and 11 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- In order to get full points, **you need your final answer to be correct and you need a justification**, unless otherwise indicated. You do not need to evaluate binomials or factorials.

Exam Format and Grading Scheme:

Answers must be written on the examination paper.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90

Students must hand in all examination materials at the end

1. (10 points) Determine the closed form expression for the n 'th term of each generating function:

(a) $\frac{3x}{1-4x} + \frac{1}{1-x}$

(b) $(3+x)^6$

(c) $\frac{1}{1+x^3}$

For each of the following sequences, find a closed form expression for the corresponding generating function (that does not involve an infinite sum). The sequences begin with index 0.

(c) $[1, 3, 5, 7, 9, \dots, 2k+1, \dots]$

(d) $[1, 4, 9, 16, 25, \dots, k^2, \dots]$

2. (10 points) Suppose there are five shows on your Netflix list. Three have 25 minute episodes, the other two have 50 minute episodes. You will watch the episodes of each show in order, but you can change shows after each episode, and watch shows in any order. Suppose you have $25n$ minutes to watch tv between studying. Let a_n be the number of ways you can watch these shows during that time.
- (a) Give a recurrence and initial conditions for a_n .

- (b) Use your recurrence and initial conditions to find a closed formula for a_n depending only on n . (If your solution to part (a) is incorrect, you will still receive marks for analysing it correctly here.)

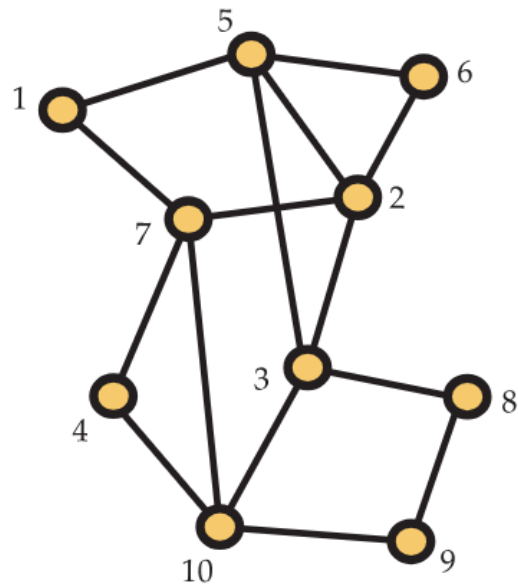
3. (10 points) Give a bijective proof of the equality

$$3^n - 2^n = \sum_{k=1}^n \binom{n}{k} 2^{n-k}$$

4. (10 points) Simone has 12 mint condition, limited edition collector cards of famous mathematical theorems (each card has a different theorem on it). She wants to display them in 4 distinct display cases. However, she can't decide how to distribute them, so she chooses a uniformly random assignment of cards, out of all the ways of distributing the cards so that each display gets at least one card.
- (a) How many different ways can Simone distribute the cards?
 - (b) What is the probability that every display gets 3 cards?
 - (c) What is the probability that one display gets 9 cards?

5. (10 points) For the following graph, determine:

- (a) if it is Eulerian;
- (b) if it is Hamiltonian;
- (c) if it is planar;
- (d) its chromatic number;
- (e) its maximum clique size.



6. (10 points) Show that there is no graph with at least 2 vertices where all the vertices have distinct degrees.

7. (10 points) A set of integers A contains an *arithmetic progression of length 3* if there exist integers a and $r \neq 0$ such that A contains $\{a, a + r, a + 2r\}$. Suppose Bob is given a set of n integers, and is able to do elementary arithmetic operations (add, multiply, subtract, divide) with numbers in the set. Approximately how many arithmetic operations does Bob need to perform in order to determine if his set contains an arithmetic progression of length 3?

8. (10 points) Suppose we choose integers a, b, c, d from the set of non-negative integer solutions to

$$a + b + c + d = 100$$

uniformly at random. What is the expected value of $a + b$?

9. (10 points) Show that the function $f(n) = 3^n - n2^n$ satisfies the recurrence $f(n) = 7f(n-1) - 16f(n-2) + 12f(n-3)$ with initial conditions $f(0) = f(1) = f(2) = 1$.