MAT344 Lecture 9

2019/June/4

1 Announcements

1. Email me about midterm conflicts
2. Midterm review materials on course homepage
3. Practice exam session on Thursday

2 This week

This week, we are talking about

1. Graphs

3 Recap

Last time we talked about

1. Complexity classes

4 The seven bridges of Königsberg

Figure 1 shows the seven bridges of the historical city of Königsberg in 1736. The citizens of the city were avid

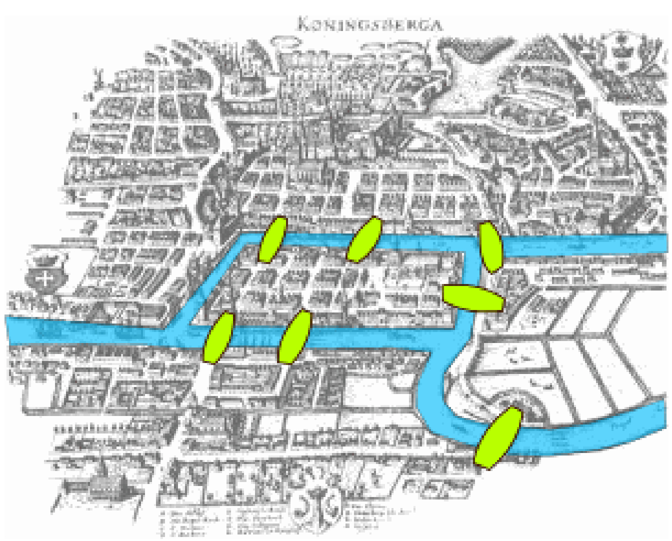

Figure 1: The seven bridges of Königsberg

walkers, and many wondered if it is possible to plan a walk that crosses every bridge of the city exactly once.

Leonhard Euler proved in 1736 that such a walk is impossible. First let's abstract away most of the irrelevant parts of the problem (see Figure 2):

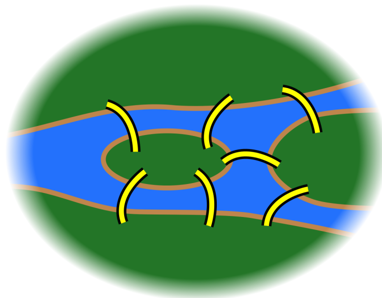

Figure 2: The seven bridges of Königsberg (sketch)

It is not important how we walk within each land mass, so we shrink these to points The structure in Figure 3

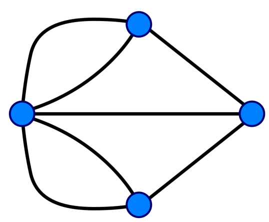

Figure 3: The seven bridges of Königsberg (graph)

is what we'll refer to as a **graph**.

Before getting to precise definitions, let's see Euler's argument for why the walk traversing all seven bridges exactly once is impossible. We'll call the points in Figure 3 **vertices** and the line segments adjacent to the vertices **edges**. What the problem is asking for is a way to traverse all edges exactly once. If, during a walk, we arrive at a vertex and leave it, we use exactly 2 of the edges adjacent to it. Therefore, any vertex visited during the walk must have an even number of edges adjacent to it, except for the start and endpoint. Since all 4 vertices have an odd number of edges adjacent to them, the walk is impossible.

5 Graphs (Chapter 5 in [KT17])

Definition 5.1. *A* ***graph*** G *is a pair* V, E *where* V *is a set and* E *is a subset of the 2-element subsets of* V*.*

Elements of V are the **vertices** and elements of E are the **edges**. Alternatively, we will use the notation $V(G)$ for the vertices and $E(G)$ for the edges of a graph G. We will often suppress the braces indicating that an edge is a 2-subset of vertices, and instead of $\{x, y\} \in E$, we will just write $xy \in E$. Often we will allow multiple edges between the same two vertices (like in the Königsberg bridge problem), the resulting structure is often called a **multigraph**. We will sometimes use the term graph to mean multigraphs, and if we want to emphasize that we are only allowing at most one edge between two vertices we will say that we are considering a **simple graph**. As in the Königsberg bridges, it is often helpful to draw a picture of a graph to try to study it. But since Definition 5.1 does not specify how one should draw this picture, we can end up with very different looking pictures of the same graph, for example, in Figure 4.

6 Graph isomorphisms

You could ask the question

Given two drawings of graphs, how do I know they are drawings of the same graph?

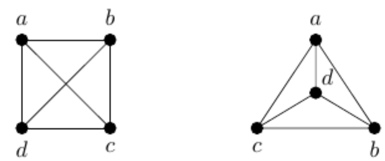

Figure 4: Two drawings of the same graph

First we should clarify what we mean by *the same graph*.

Definition 6.1. *If $G = (V, E)$ and $H = (W, F)$ are graphs, then we say that G is* ***isomorphic*** *to H and write $G \cong H$ if there exists a bijection $f : V \to W$ such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in F$.*

The question to decide whether two given finite graphs are isomorphic is known as the **Graph isomorphism problem** and it is difficult! The best algorithm to decide graph isomorphism had complexity $O(2^{\sqrt{n \log n}})$ (where n is the number of vertices of the graph), until 2017, when a new algorigthm was announced with complexity $O(2^{\left((\log n)^3\right)})$, which is quite close to being polynomial time.

Some graphs are easy to identify so they get special names:

Definition 6.2. *The* ***complete graph*** *K_n on n vertices is a graph where xy is an edge for all $x, y \in [n]$. The* ***independent graph*** *I_n is a graph on n vertices that has no edges.*

Exercise 6.3. *Show that the two graphs in Figure 5 are isomorphic.*

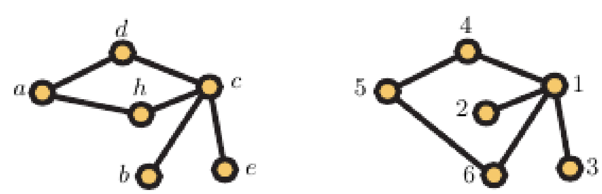

Figure 5: Two isomorphic graphs

Solution: The isomorphism is given by

$$f(a) = 5, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 6, \quad f(e) = 2, \quad f(h) = 4.$$

In many cases, it is easy to see when two graphs *are not* isomorphic.

Exercise 6.4. *Explain why the two graphs in Figure 6 are not isomorphic.*

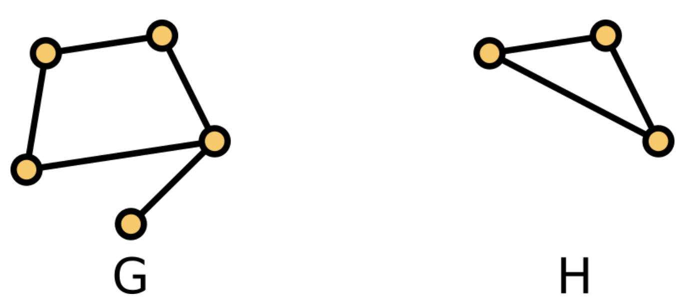

Figure 6: Two nonisomorphic graphs

Solution: The graph G has 5 vertices and H has only 3. Isomorphic graphs must have the same number of vertices.

Definition 6.5. *If $G = (V, E)$ and $H = (W, F)$ are graphs we say H is a* ***subgraph*** *of G when $W \subseteq V$ and $F \subseteq E$. We say H is an* ***induced subgraph*** *of G when $W \subseteq V$ and $F = \{\{x, y\} \in E \mid x, y \in W\}$ (see Figure 7 for an example).*

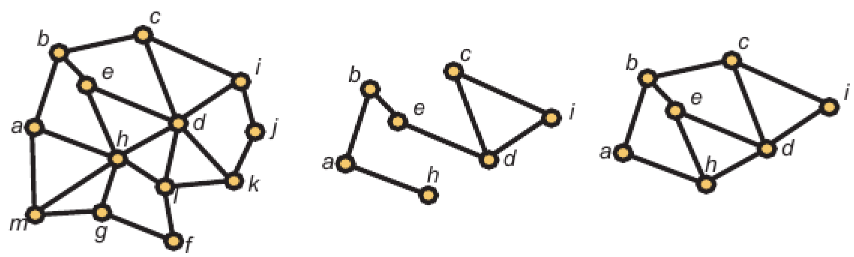

Figure 7: A graph, a subgraph and an induced subgraph

Given two graphs H and G, determining if H is a subgraph of G is a really hard problem with no efficient algorithms (no polynomial time algorithms known).

Definition 6.6. *A sequence $(x_1, \ldots, x_n)$ of vertices in a graph is called a* ***walk*** *when $x_i x_{i+1}$ is an edge. If the vertices in a walk are all distinct, then the walk is called a* ***path****. When $n \geq 3$, a path $(x_1, \ldots, x_n)$ is called a* ***cycle*** *if $x_n x_1$ is also an edge.*

Exercise 6.7. *Explain why the two graphs in Figure 8 are not isomorphic.*

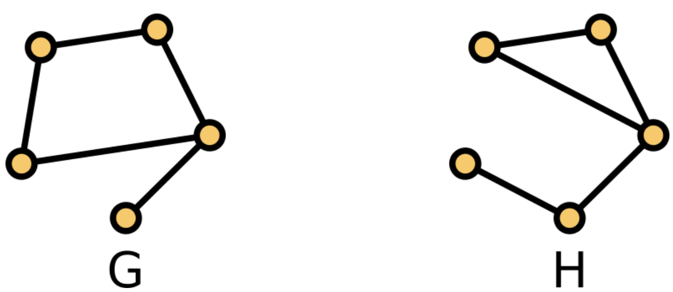

Figure 8: Two nonisomorphic graphs

Solution: The graph G contains a 4-cycle as a subgraph, and H does not.

Exercise 6.8. *Explain why the two graphs in Figure 9 are not isomorphic.*

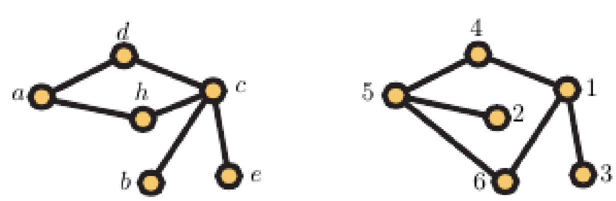

Figure 9: Two nonisomorphic graphs

Solution: The vertex c is adjacent to four other vertices in the first graph. There is no vertex in the second graph adjacent to four other vertices.

This leads us to the following definition:

Definition 6.9. *The* ***degree*** *of a vertex v in a graph G, denoted $\deg_G(v)$ is the number of edges incident to it.*

We could now restate the argument we gave in the solution to Exercise 6.8 like this: For the first graph, $\deg(c) = 4$ and the second graph has no vertex of degree 4.

Theorem 6.10. *Degree is invariant under isomorphism. That is, if $f : G \to H$ is a graph isomorphism, then for any vertex $x \in V(G)$, we have*

$$\deg_G(x) = \deg_H(f(x)).$$

Proof. The number $\deg_G(x)$ is the number of elements in the set of vertices adjacent to x. Let $A = \{y \in V(G) \mid xy \in E(G)\}$. Since f is an isomorphism, we have

$$\{f(y) \in V(H) \mid y \in A\} = \{y' \in V(H) \mid f(x)y' \in V(H)\},$$

therefore

$$\deg_H(f(x)) = |\{y' \in V(H) \mid f(x)y' \in E(H)\}| = |f(A)| = |A| = \deg_G(x).$$

Q.E.D.

This leads us to an easy to check criterion for graph isomorphism

Theorem 6.11. *Two isomorphic graphs must have the same degree sequence. That is, if we list the sequence of the degrees of the vertices of the two graphs in (weakly) decreasing order, the two sequences must be the same.*

Proof. Under a graph isomorphism, any vertex must be mapped to a vertex with the same degree.

Q.E.D.

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 2