MAT344 Lecture 5

2019/May/20

1 This week

This week, we are talking about

1. Recursion
2. Induction

2 Recap

Last time we talked about

1. Lattice paths
2. The binomial theorem

3 Recursion (Chapter 3.4 in [KT17])

Let's find a formula for the number of ways of triangulating a convex polygon. We start by computing some small examples. We see that In addition to the 1 way of triangulating the triangle, there are 2 ways to triangulate a

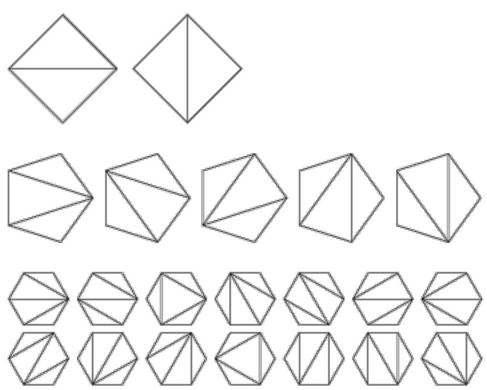

Figure 1: Triangulations of convex polygons

square, 5 to triangulate a pentagon and 14 to triangulate the hexagon. These agree with the number of Dyck paths for $n = 1, 2, 3, 4$. We would like to prove this, but there does not seem to be an obvious combinatorial proof.

Let us try a different method. We will find a way to express C_n in terms of C_ks with $k \leq n$. This is known as a **recurrence relation**. Then if any other counting problem satisfies the same recurrence relation and agrees with our values in small examples, the answer must be C_n.

Let us clarify what we mean here. Recall that we defined factorials as

$$n! = n(n-1)(n-2)\dots 2 \cdot 1.$$

Technically, this is not quite a flawless definition, since multiplication should be a *binary operation*, i.e. you are only supposed to multiply two numbers at a time. We could instead say

$$n! = n \cdot (n-1)!.$$

It seems like that we are just pushing the problem one step further but if we also define $0! = 1$, then we see that (after n steps), we can find the value of $n!$. The formula $n! = n \cdot (n-1)!$ that lets us compute the value of a function (the one sending a number n to $n!$) in terms of other values of the function is called a **recursive formula**. We call this process **recursion**.

Recursion can involve more than one variable, for example, we proved the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{1}$$

with binomial coefficients. If we define $\binom{n}{n} = \binom{n}{0} = 1$ for all n, then we can use (1) to compute binomial coefficients.

Exercise 3.1. *What does this recursion look like in Pascal's triangle?*

Let us get back to Catalan numbers that we know count Dyck paths. Notice that after the first up step, any Dyck path will touch the diagonal at some point. Let (k, k) be the first time this happens. Then the $2k$-th step must have been a up step. Since this is the first time we are touching the diagonal, our path from $(1, 0)$ to $(k, k-1)$ never crosses the line $y = x - 1$, so we may remove the first and last step from this initial segment and end up with a Dyck path from $(0, 0)$ to $(k-1, k-1)$, in addition to the tail of the path (from (k, k) to (n, n)), which itself can be considered as a Dyck path from $(0, 0)$ to $(n-k, n-k)$. Counting Dyck paths this way we obtain the recursive formula

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

It is customary to reindex this as

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}. \tag{2}$$

Let's check that this agrees with what we know about Catalan numbers. We define $C_0 = 1$ and we compute

$$\begin{aligned}
C_0 &= 1 \\
C_1 &= C_0 C_0 = 1 \\
C_2 &= C_0 C_1 + C_1 C_0 = 2 \\
C_3 &= C_0 C_2 + C_1 C_1 + C_2 C_0 = 5 \\
C_4 &= C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0 = 14
\end{aligned}$$

We want to show that triangulations of polygons satisfy the same recurrence. The relation suggests that from a triangulation of an $n+2$-gon, we should produce two other polygons (with triangulations), a k-gon, and an $n-k+1$-gon, since we expect $j+2$-gons to be counted by C_j. Number the vertices from 1 to $n+2$ and focus on the single external edge between the two vertices 1 and $n+2$. This is part of a triangle in the triangulation, and it connects to, say, vertex k. The rest of the polygon is now split into two polygons (with triangulations). Note that if $k = 2$ or $k = n+1$ one of these polygons is empty, and we have just removed a triangle from our $n+2$-gon to get a triangulation of an $n+1$-gon. If $3 \leq k \leq n$, then we get a k-gon on one side and an $n-k+1$-gon on the other side (both with triangulations).To check that this is a bijection, note that we can put the triangulations of the k-gon and $n-k+1$-gon together with the triangle connecting vertices $1, k$, and $n+2$ to recover the triangulation of the $n+2$-gon. This shows that the number of triangulations of a convex polygon satisfy the same recurrence, and therefore we conclude that the number of triangulations of a convex $n+2$-gon is C_n.

References

[Gui18] David Guichard. *Combinatorics and Graph Theory*. Open access, 2018. Available at https://www.whitman.edu/mathematics/cgt_online/book/. 3

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 3