MAT344 Lecture 3

2019/May/14

1 Announcements

1. How is the book?
2. Are the introductions/discussions helpful?
3. The Quercus discussions are great!
4. HW due date?

2 This week

This week, we are talking about

1. Binomial coefficients
2. Lattice paths and Catalan numbers
3. The binomial theorem

3 Recap

Last time we talked about

1. Combinations
2. Combinatorial proofs

4 Binomial coefficients (Chapter 2.5 in [KT17])

We continue looking at counting problems that we can solve with binomial coefficients.

Exercise 4.1 ([KT17], Example 2.18)**.** *The office assistant is distributing supplies. In how many ways can he distribute 18 identical folders among four office employees: Audrey, Bart, Cecilia and Darren, with the additional restriction that each will receive at least one folder?*

Solution: Imagine the folders placed in a row. Then there are 17 gaps between them. Of these gaps, choose three and place a divider in each. Then this choice divides the folders into four non-empty sets. The first goes to Audrey, the second to Bart, etc. Thus the answer is $\binom{17}{3}$. Figure 1 illustrates the distribution where Audrey gets 6, Bart 1, Cecilia 4 and Darren 7 folders.

This sort argument is commonly referred to as a **stars and bars** computation. We have a row of n indistinguishable stars (in the folders example, $n = 18$, but let us consider the case $n = 7$ here)

$$* * * * * * *$$

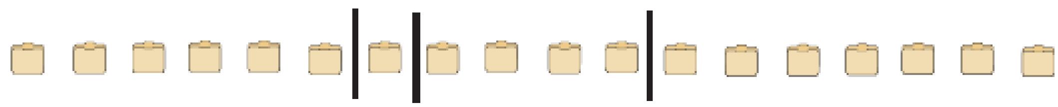

Figure 1: One way of distributing the folders

that we want to separate into k piles. We do this by inserting $k-1$ bars into the ($n-1$ many) spaces between the stars (in the folder example and the stars and bars example, $k=4$)

$$*|**|**|**$$

so we have reduced the counting problem to a previous one that we solved already, with answer $\binom{n-1}{k-1}$.

Exercise 4.2. *Suppose we redo the preceding problem but drop the restriction that each of the four employees gets at least one folder. Now in how many ways can we distribute the folders?*

Solution: How could we reduce this to the previous problem? If we had 22 instead of 18 folders, we could distribute them using the previous method (in $\binom{21}{3}$ many ways) so everyone would have at least one folder. Then we could take one folder from everyone. Let us go back to the bars and stars problem above, with $n=7$ and $k=4$, but this time allowing empty piles. How could we think of this? Eventually we want to have all 7 stars and 3 bars placed in a row, so there are 10 "places" where either a star or a bar can go. If we know where the bars go, the stars will fill all the remaining places, so there are $\binom{10}{3}$ many ways of doing this.

So there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ many ways of separating n indistinguishable stars in to k piles, some of which may be empty.

The following problem does not seem to be immediately related to the bars and stars problem, but we will soon find out that it is:

Exercise 4.3 ([Mor17], Example 5.2)**.** *Chris has promised to bring back donuts for three friends he's studying with. He wants to buy 8 donuts, and the donut shop sells five varieties. How many ways are there for Chris to fill this order?*

Solution: Let us number the flavors as 1, 2, 3, 4, and 5. Chris wants to order 8 donuts, which we can represent as a [5]-string of length 8. We only care about how many times a letter appears in the string, and not *where* it appears, so we may assume that the numbers appear in increasing order, for example

$$111334444$$

is an order where Chris buys 3 bagels with flavor 1, no bagels with flavor 2, 2 bagels with flavor 3, 4 with flavor 4 and none with flavor 5. If we suggestively put separating lines between the different flavors, we get a picture that looks something like this

$$111||33|4444|$$

(notice the two separators between the 1s and the 3s, signifying that there are no 2s). If we know where the separators are, we can recover how many of each number we have, so we don't really need to write all the numbers out, and we end up with a picture

$$***||**|****|$$

and this is a stars and bars computation, whose solution we already know to be $\binom{8+5-1}{5-1} = \binom{12}{4}$.

This counting problem where we are choosing k objects from n, but we are allowed to select a single object more than once is called a **combination with repetition**.

Theorem 4.4 ([Mor17], Theorem 5.3)**.** *The number of ways of choosing k objects from n objects, with repetition allowed is*

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Proof. We use the idea from the previous problem. We assume that we have an inexhaustible supply of each letter in $[n]$. Since there are n different types of objects, we need $n-1$ bars and k stars. So we need to select k positions from the $n+k-1$ places.

Q.E.D.

We can also reformulate these computations in terms of integer solutions of inequalities.

Exercise 4.5 ([KT17], Example 2.21.)**.** *We count the number of integer solutions to the inequality*

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

subject to various conditions on the values of $x_1, x_2, \ldots, x_6$. For each of the following cases, find the number of solutions in terms of a binomial coefficient.

1. *When all $x_i > 0$ and equality holds.*

2. *When all $x_i \geq 0$ and equality holds.*

3. *When $x_1, x_2, x_4, x_6 > 0, x_3 = 52, x_5 = 194$ and equality holds.*

4. *When all $x_i > 0$ and the inequality is strict.* ***Hint:*** *Introduce a new variable x_7 which is the balance. Note that x_7 must be positive.*

5. *When all $x_i \geq 0$ and the inequality is strict.* ***Hint:*** *Add a new variable x_7 as above. Now it is the only one which is required to be positive.*

6. *When all $x_i \geq 0$.*

Let us recast all the typical counting problems we have seen so far in terms of strings. In each case, we are going to be counting $[n]$-strings of length k. We have two decisions to make. We should decide if the order in which the letters appear in the string matter and we have to decide whether we are allowed to repeat letters from the alphabet. We can summarize our findings in Table 1 ([Mor17], Table 5.1)

	repetition allowed	repetition not allowed
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k}$	$\binom{n}{k}$

Table 1: number of $[n]$-strings of length k

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 3

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 2, 3