MAT344 Lecture 23

2019/Aug/6

1 Announcements

2 This week

This week, we are talking about

1. The max flow/min cut theorem
2. The Ford-Fulkerson algorithm

3 Recap

Last time we talked about

1. Networks

4 Network flows (Ch. 13 in [KT17])

Recall that a **flow** in a network is a function f which assigns to each directed edge $e = (x, y)$ a non-negative value $f(e) = f(x, y) \leq c(e)$ such that for every vertex which is neither the source nor the sink, we have conservation of flow, i.e. the amount leaving y is equal to the amount entering y.

Flows clearly exist in any network, we can just let $f(e) = 0$ for any edge, but this is not a particularly useful flow in applications. One of our objectives is to find a flow of maximum strength through a network. What are some upper bounds we can find for the strength of a flow? Two easy upper bounds are the sum of the capacities of all edges leaving the source, and the sum of the capacities entering the sink.

Definition 4.1. *A **cut** in a network is a pair (X, Y) of subsets of the vertex set V such that $X \cup Y = V, s \in X, t \in Y$. The **capacity** $c(X, Y)$ of the cut is the sum of the capacities of the edges directed from X to Y (i.e. edges $e = (x, y)$ with $x \in X$ and $y \in Y$.*

Figure 1 shows a cut in a network.

Proposition 4.2. *The capacity of any cut is an upper bound for the strength of any flow. Moreover, the strength of a flow f can be computed as $f(X, Y) - f(Y, X)$, where $f(A, B)$ denotes the sum of the values of f on all edges directed from A to B.*

Proof. Let f be a flow and let $V = X \cup Y$ be a cut. Since there is no inflow to the source, we have $\sum_{z \in V} f(z, s) = 0$, and the strength of the flow is $|f| = \sum_{y \in V} f(s, y)$. Also, by the conservation laws, for $x \neq s, t$,

$$\sum_{y \in V} f(x, y) - \sum_{z \in V} f(z, x) = 0.$$

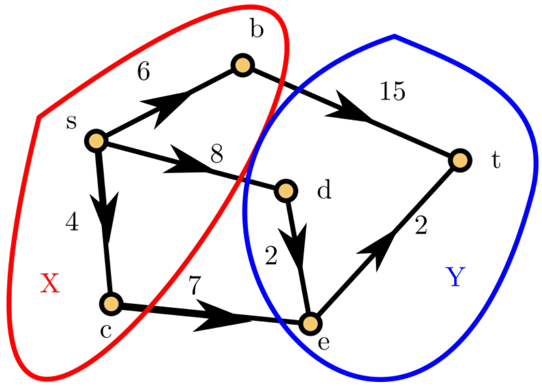

Figure 1: A cut in a network

We compute

$$\begin{aligned} |f| &= \sum_{y\in V} f(s,y) - \sum_{z\in V} f(z,s) \\ &= \sum_{y\in V} f(s,y) - \sum_{z\in V} f(z,s) + \sum_{\substack{x\in X\\ x\neq s}} \left[\sum_{y\in V} f(x,y) - \sum_{z\in V} f(z,x) \right] \\ &= \sum_{x\in X} \left[\sum_{y\in V} f(x,y) - \sum_{z\in V} f(z,x) \right] \end{aligned}$$

Notice that if (x,w) is an arc with both endpoints in X, then we get a cancellation $f(x,w) - f(x,w)$ as we are computing the sum on the last line, since we are summing over all $x \in X$. If $x \in X$ and $y \in Y$, then we get a contribution of $f(x,y)$. Therefore

$$|f| = \sum_{x\in X, y\in Y} f(x,y) - \sum_{x\in X, y\in Y} f(y,x) \leq \sum_{x\in x, y\in Y} f(x,y) \leq \sum_{x\in X, y\in Y} c(x,y) = c(X,Y).$$

Q.E.D.

So the maximum value of the flow is bounded above by any cut, in particular, a cut of minimal capacity. In the network we are considering, the cut in Figure 2 is minimal.

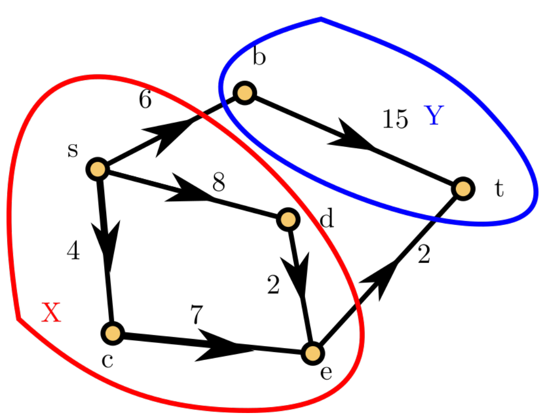

Figure 2: A minimal cut with $c(X,Y) = 8$

And Figure 3 shows a flow with $|f| = 8$.

In this case, we know that the flow in Figure 3 is maximal, because it's equal to the upper bound (the value of the minimal cut).

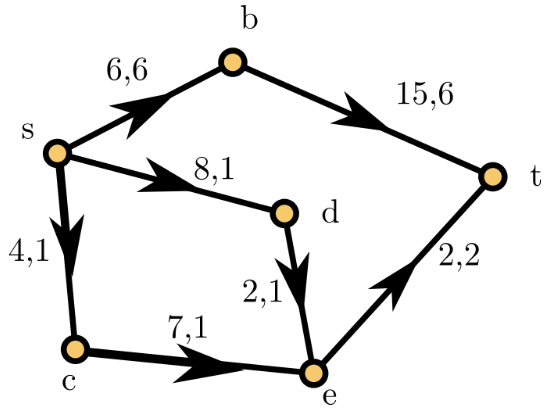

Figure 3: A flow with $|f| = 8$

Theorem 4.3 (Ford-Fulkerson, Theorem 13.8 in [KT17])**.** *Let $G = (V, E)$ be a network. Then let v_0 be the maximum value of a flow, and let c_0 be the minimum capacity of a cut. Then*

$$v_0 = c_0.$$

Definition 4.4. *Let f be a flow in our network. We will say that the sequence $P = x_0, x_1, \ldots, x_k$ is a* ***special path****, or* ***augmenting path*** *from x_0 to x_k if for each i, $1 \leq i \leq k$, either*

1. *$e = (x_{i-1}, x_i)$ is an edge with $c(e) - f(e) > 0$, or*
2. *$e = (x_i, x_{i-1})$ is an edge with $f(e) > 0$.*

Consider the network on Figure 4

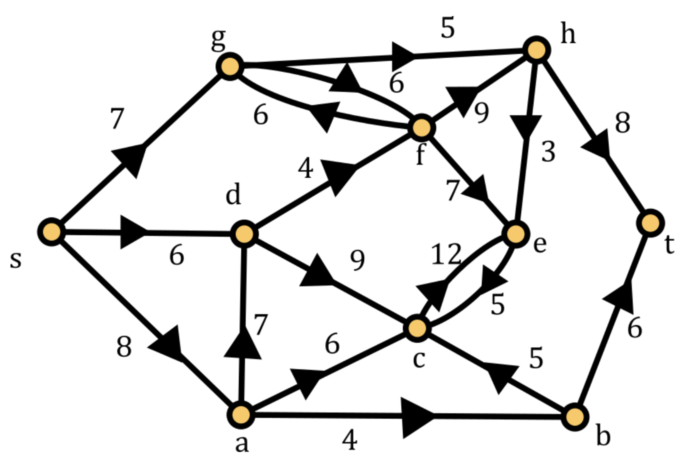

Figure 4: A network

Edges e with $f(e) = c(e)$ are said to be *saturated* and the two conditions above can be stated in words as requiring the 'forward' edges of the path to be unsaturated while requiring 'backward' edges to have positive flow.

For example, on the flow in Figure 5, the path in red is an augmenting path.

Notice that we can increase the flow strength if, for all 'forward' edges in the augmenting path, we increase the flow value by 1, and for the 'backward' edge, we decrease the flow value by 1 to get the flow on Figure 6

For an augmenting path, we define two quantities

$$\delta_1 = \min\{c(x_{i-1}, x_i) - f(x_{i-1}, x_i) | (x_{i-1}, x_i) \text{ is a 'forward' edge of } P\}$$
$$\delta_2 = \min\{f(x_i, x_{i-1}) | (x_{i-1}, x_i) \text{ is a 'forward' edge of } P\}$$

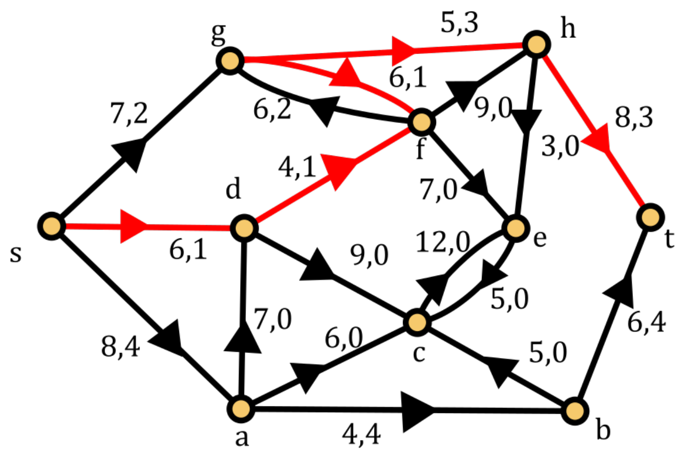

Figure 5: A flow and an augmenting path

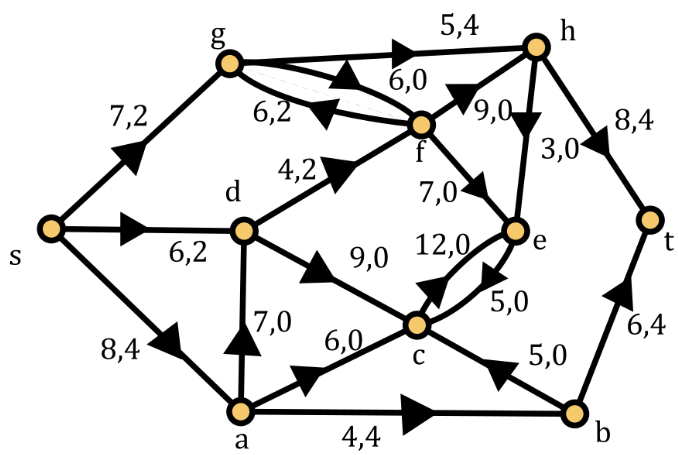

Figure 6: Augmenting the flow

Then we let $\delta = \min\{\delta_1, \delta_2\}$. What does this δ represent? This is the amount of flow that can be 'pushed through' the augmenting path. We do the same thing as in the example, we increase the flow by δ at the 'forward' edges and decrease at the 'backward' ones. This new assignment is then also a flow.

Suppose there exists an augmenting path P from s to t. Then we can increase the total flow value by δ.

What if for a flow, there is no augmenting path from s to t? We will produce a cut in our network the following way: Let X be the set of vertices that can be reached from s using an augmenting path, and let Y be the set of remaining vertices. Clearly $s \in X$, $t \in Y$ and $X \cup Y = V$, so this is indeed a cut.

Notice that all edges $e = (x, y)$ with $x \in X, y \in Y$ must be saturated (at full capacity) because otherwise we could reach y from s using an augmenting path. Also, all edges $e(y, x)$ with $y \in Y, x \in X$ must have $f(e) = 0$ (same argument). Therefore we have (using Proposition 4.2)

$$|f| = f(X, Y) - f(Y, X) = c(X, Y).$$

Notice what this equality implies. Since the capacity of any cut is an upper bound for the strength of any flow, and we have a flow whose strength is equal to a capacity of a cut, the cut we constructed must be minimal, and the flow we had must have been maximal. This proves Theorem 4.3.

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 3