MAT344 Lecture 2

2019/May/9

1 Announcements

2 This week

This week, we are talking about

1. Basic counting techniques,
2. Permutations,
3. Combinations

3 Recap

Last time we talked about

1. Counting strings
2. Counting permutations

4 Combinations (Chapter 2.3 in [KT17])

The next problem illustrates that we can solve many problems with the basic principles from last lecture, but we may be better off looking for a more general formula.

Exercise 4.1 ([Bog04], Chapter 1.2 Problem 2)**.** *Some n schools are going to send their baseball teams to a tournament, and each team must play each other team exactly once. Let us think of the teams as numbered 1 through n.*

1. *How many games does team 1 have to play in?*
2. *How many games, other than the one with team 1, does team two have to play in?*
3. *How many games, other than those with the first $i-1$ teams does team i have to play in?*
4. *In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?*

The natural form of the answer to part 4. above is $\sum_{k=1}^{n}(n-k)$. This raises the question: What should be considered an answer to a counting problem? The answers we gave last time (e.g. $\frac{m!}{(m-n)!}$ for the number of permutations of length n from a set with m elements) were what we call "closed formulas", i.e. formulas that do not involve iterating over a set[1], in particular, they do not involve summation signs. On the other extreme there is always the completely useless formula $|X| = \sum_{x \in X} 1$ that requires listing all the elements of X.

[1]The way we defined factorials, this actually involves iterating over a set and taking product, but we consider factorials to be a very basic object in combinatorics and we are happy with answers that are given in terms of factorials

The formula $\sum_{k=1}^{n}(n-k)$ is between the two extremes. It is certainly better than $\sum_{x\in \text{ games}} 1$, but maybe we can improve on it to not include a summation sign. Let's interpret the question in terms of strings. We are looking for $[n]$-strings of length 2 with no repeats, so there should be $P(n,2)$ many of them. However, for any $1 \leq i < j \leq n$, we counted both (i,j) and (j,i) and we want each team to play every other team exactly once (there is no "home team"), so we overcounted by a factor of 2, so altogether there are $\frac{P(n,2)}{2} = \frac{n!}{2(n-2)!}$ many matches. This is a better answer than the one we gave before.

Exercise 4.2 ([Bog04], Chapter 1.2.6 Problem 34)**.** *Consider the ice cream parlor with* 12 *different flavors and triple-decker cones from the previous lecture (recall that having chocolate as the bottom scoop is different from having it on top). A person claims that to order* 3 *different flavors here is the same as choosing* 3 *different flavors.*

1. *What is the difference between the number of ways to stack ice cream in a triple decker cone with three different flavors and the number of ways to simply choose three different flavors? What aspect of the triple decker cone is this person neglecting?*
2. *In particular, how many different triple decker cones use chocolate, strawberry and vanilla?*
3. *Using your answer from the previous part, compute the number of ways to choose three different flavors of ice cream from the number of ways to choose a triple-decker cone with three different flavors.*

More abstractly, in problem 4.1 and problem 4.2, we wanted to enumerate subsets of a certain size of a given set.

Definition 4.3. *If X is a set with $|X| = n$ and $0 \leq k \leq n$, then the number of k-element subsets of X is the **binomial coefficient** $\binom{n}{k}$ or alternatively $C(n,k)$ (read: "n choose k"). We also call a k-element subset of X a k-combination of X.*

We can express binomial coefficients in terms of factorials.

Proposition 4.4 ([KT17], Proposition 2.8)**.** *If $0 \leq k \leq n$ are integers, then the number of k-combinations of a set with n elements is*

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Proof. Let X be an n-element set. The number of k-permutations of X is $P(n,k)$. We have overcounted each combination by a factor of $k!$, since the order in which we pick them does not matter.

Q.E.D.

5 Combinatorial Proofs (Chapter 2.4 in [KT17])

Like many concepts, combinatorial proofs are best explained by an example. Consider the following identity, for $0 \leq k \leq n$:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{1}$$

To prove this identity, your first instinct may be to expand everything to factorials and manipulate the algebra.

Proof. The right hand side can be expressed as

$$\begin{aligned}\binom{n-1}{k}+\binom{n-1}{k-1} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\ &= \frac{k(n-1)!}{k(k-1)!(n-k)!}+\frac{(n-k)(n-1)!}{k!(n-k)(n-k-1)!}\\ &= \frac{k(n-1)!}{k!(n-k)!}+\frac{(n-k)(n-1)!}{k!(n-k)!}\\ &= \frac{(n-k+k)(n-1)!}{k!(n-k)!}\\ &= \frac{n!}{k!(n-k)!}\\ &= \binom{n}{k},\end{aligned}$$

which is the left hand side, so the two sides are equal.

Q.E.D.

This is certainly a proof, but it does not seem like it offers much in terms of an *explanation*. We have gotten from one side to the other by essentially moving symbols around.

Now compare this above proof to the following argument, which we'll call a combinatorial proof.

Proof. The LHS is counting k-element subsets of $[n]$. For any subset S of size k, either $n \in S$ or $n \notin S$. In the first case, the remaining elements of S form a subset of $[n-1]$ of size $k-1$, and in the second case, the remaining elements of S form a subset of $[n-1]$ of size k. There are $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ of these, respectively.

Q.E.D.

How is this proof different from the algebraic one? We *interpreted* the numbers of the LHS and RHS as sizes of two sets, namely, subsets of $[n]$ of size k on the LHS and subsets of $[n-1]$ of size $k-1$ and subsets of $[n-1]$ of size k on the RHS, and we showed that there is a bijection between these two sets. Proving something by constructing an explicit bijection between two sets that have the same size is what we will refer to as a **combinatorial**, or **bijective proof**.

Richard Stanley, possibly the most famous combinatorialist alive, after introducing a non-combinatorial and a combinatorial proof of a result, says the following ([Sta97], p22):

"Not only is the above combinatorial proof much shorter than our previous proof, but it also makes the reason for the simple answer completely transparent. It is often the case, as occurred here, that the first proof to come to mind turns out to be laborious and inelegant, but that the final answer suggests a simpler combinatorial proof."

Now we'll look at a few combinatorial proofs:

1. Let n be a positive integer and $0 \le k \le n$. Explain why

$$\binom{n}{n-k}=\binom{n}{k}.$$

2. ([KT17], Example 2.12) Let n be a positive integer. Using Figure 1, explain why

$$\sum_{i=1}^{n} i=\binom{n+1}{2}.$$

3. ([KT17], Example 2.13) Let n be a positive integer. Using Figure 2, explain why

$$\sum_{i=1}^{n}(2i-1)=n^2.$$

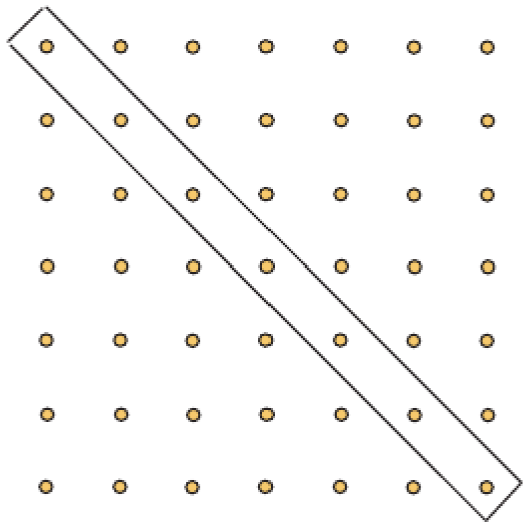

Figure 1: The sum of the first n integers

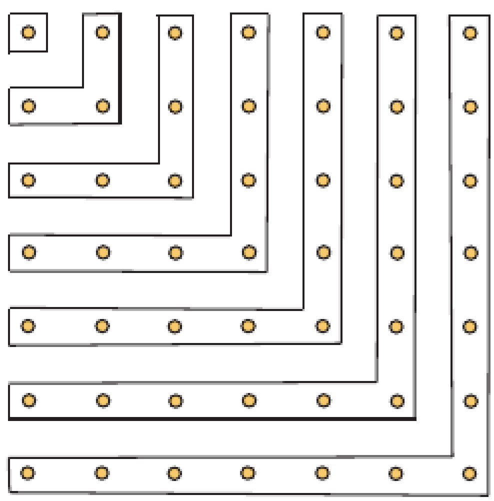

Figure 2: The sum of the first n odd integers

4. ([KT17], Example 2.14) Let n be a positive integer. Explain why

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

   Hint: Think of subsets of $[n]$.

5. ([KT17], Example 2.16) Explain the identity

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i$$

   Hint: Think of both sides as counting ternary strings (that is, [3]-strings) of length n.

6. ([KT17], Example 2.17) For each non-negative integer n,

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

Hint: Think of both sides counting bit strings of length $2n$ with half the bits being 0's.

7. ([Sta97], 1.1.17 Example) Let a, b, n be nonnegative integers. Verify the identity

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}.$$

Hint: Think of both sides counting n-element subsets of $[a+b]$.

For some excellent problems on finding combinatorial proofs for identities, see Exercise 3 after Chapter 1 of [Sta97], or http://www-math.mit.edu/~rstan/bij.pdf.

6 Binomial coefficients (Chapter 2.5 in [KT17])

Recall the result about binomial coefficients that we just proved (1):

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Notice that this equation expresses one binomial coefficient as the sum of two other binomial coefficients, with $n-1$ instead of n in the "top spot". This relation is the basis for Pascal's triangle (Figure 3).

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Figure 3: Pascal's triangle

For some fun problems on Pascal's triangle, see Chapter 1.2.5 of [Bog04], we only mention one of them here: can you find a way of interpreting the equation

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n?$$

Exercise 6.1 ([Lev18], Chapter 1.4)**.** *The Stanley Cup is decided in a best of seven tournament between two teams. In how many ways can your team win? Let's answer this question two ways:*

1. *How many of the* 7 *games does your team need to win? How many ways can this happen?*
2. *What if the tournament goes all* 7 *games? So you win the last game. How many ways can the first* 6 *games go down?*
3. *What if the tournament goes just* 6 *games? How many ways can this happen? What about* 5 *games?* 4 *games?*
4. *What are the two different ways to compute the number of ways your team can win? Write down an equation involving binomial coefficients. What pattern in Pascal's triangle is this an example of?*
5. *Generalize. What if the rules changed and you played a best of* 9 *tournament* (5 *wins required)? What if you played an n game tournament?*

Solution:

1. We need to win 4 games out of 7, and this can happen in $\binom{7}{4}$ many ways.
2. If it comes to a game 7, then our team has won 3 our of the first 6 games. There are $\binom{6}{3}$ many ways for this.
3. Similarly to the previous part if the tournament goes to $6, 5$, or 4 games, there are $\binom{5}{3}, \binom{4}{3}, \binom{3}{3}$ many ways, respectively.
4. Comparing the two ways of counting, we find that

$$\binom{7}{4} = \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3}.$$

Exercise 6.2 ([Lev18], Chapter 1.4, Exercise 4.)**.** *A woman is getting married. She has 15 best friends but can only select 6 of them to be her bridesmaids, one of which needs to be her maid of honor. How many ways can she do this?*

1. *What if she first selects the 6 bridesmaids, and then selects one of them to be the maid of honor?*
2. *What if she first selects her maid of honor, and then 5 other bridemaids?*
3. *Explain why* $6\binom{15}{6} = 15\binom{14}{5}$.
4. *Generalize. What if she has n best friends, only k of whom can be bridesmaids, and only j of them can be maids of honor? What is the resulting binomial identity?*
5. *Generalize further. What if we have an n-element set, and we have to choose a k_1-element subset of it, and a k_2-element subset of that subset, and so on until a final j-th subset of size k_j (of course $n \geq k_1 \geq k_2 \ldots \geq k_j$). What is the resulting binomial identity?*

References

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[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics.* Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2, 3, 4, 5

[Lev18] Oscar Levin. *Discrete Mathematics, an Open Introduction.* Open access, 2018. Available at http://discrete.openmathbooks.org. 5, 6

[Sta97] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997. Available at http://www-math.mit.edu/~rstan/ec/ec1.pdf. 3, 5