MAT344 Lecture 19

2019/July/23

1 Announcements

2 This week

This week, we are talking about

1. Probability

3 Recap

Last time we talked about

1. Recurrences

4 Probability (Ch. 10 in [KT17])

Example 4.1. *We roll two fair six-sided dice (a red one and a blue one) and add up the result. In how many ways can we roll a total of n?*

This is equivalent to solving

$$x_1 + x_2 = n$$

for $1 \leq x_1, x_2, \leq 6$. Using inclusion-exclusion, we find that the number of solutions is

$$\binom{2}{0}\binom{n-1}{1} - \binom{2}{1}\binom{\max(n-7,0)}{1} + \binom{\max(n-13,0)}{1}.$$

Note that we have to use the maximum functions, since we extended the definition of binomial coefficients to negative numbers, so, for example, $\binom{-3}{1} = -3$. We can write these in a table

value of n	2	3	4	5	6	7	8	9	10	11	12
number of ways	1	2	3	4	5	6	5	4	3	2	1

(the result is 0 for all other values of n).

Since these are all the possible outcomes of the dicerolls, and if the dice are fair, then all outcomes are equally likely, we could rephrase the solution to the above problem in terms of *probabilities*. Instead of the number of ways a certain outcome can happen, we will record the number of ways it can happen *divided by the total number of possible outcomes*

value of n	2	3	4	5	6	7	8	9	10	11	12
probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Definition 4.2. *A* ***probability space*** *is a pair (S, P) where S is a finite set and P is a function whose domain is the set of all subsets of S and whose range is the set $[0,1] \subset \mathbb{R}$, satisfying the following:*

1. *$P(\emptyset) = 0$ and $P(S) = 1$.*

2. *If A and B are subsets of S, and $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.*

The function P is called a ***probability measure****. The subsets of S are called* ***events*** *and $P(E)$ is the* ***probability*** *of E. A single element $s \in S$ is commonly referred to as an* ***outcome****.*

In example 4.1, the set S is all possible outcomes of the dicerolls. We can formalize this as ordered pairs (i, j) with $1 \leq i, j \leq 6$. One event E is the situation when the sum of the dicerolls is, for example, 7. We already computed that $P(E) = \frac{6}{36} = \frac{1}{6}$. The fact that the two dice are fair mean that $P(\{(i, j)\} = \frac{1}{36}$ for all pairs (i, j).

5 Conditional Probability and Independent Events (Ch. 10.2 in [KT17])

Consider the situation of Example 4.1, but what if we roll the red die first and it comes up as 3? What can be said about the probabilities of the sum of the two dicerolls now? For example, now it's impossible for us to roll a 3 or a 10 (among other things), but the probabilities of the possible outcomes should also be affected. Since the blue die is also fair, it has an equal probability to come up k for $1 \leq k \leq 6$, so the probabilities of the two rolls are now

value of n	2	3	4	5	6	7	8	9	10	11	12
probability	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0

Definition 5.1. *Let (S, P) be a probability space and let B be an event for which $P(B) > 0$. Then for every event $A \subset S$, we define the* ***probability of A, given B****, denoted $P(A|B)$ by setting*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

In the above example, if we let B to be the event that the red die comes up 3, and A the even that the two rolls come up to a total of n. The probability $P(A \cap B)$ is equal to $\frac{1}{36}$ if n is between 4 and 9 and 0 otherwise.

Exercise 5.2 (Example 10.8. in [KT17]). *There are two jars of marbles. The first one has six red, nine blue and five green. the second one has nine red, five blue and four green. A jar is selected at random and from this jar, two marbles are chosen at random. What is the probability that both are green?*

Solution: Let G be the event that both marbles are green. Let J_1 be the event that the first and J_2 be the event that the second jar is chosen. Then $G = (G \cap J_1) \cup (G \cap J_2)$ since $J_1 \cup J_2 = S$. Also, $G = (G \cap J_1) \cap (G \cap J_2) = \emptyset$, since either the first or second jar is chosen in all outcomes. Therefore

$$P(G) = P(G \cap J_1) + P(G \cap J_2).$$

How do we compute $P(G \cap J_1)$ an $P(G \cap J_1)$? We can compute the probabilities $P(G|J_1)$ and $P(G|J_2)$ more easily, since these are just probabilities of picking two greens from a single jar. We have

$$P(G|J_1) = \frac{\binom{5}{2}}{\binom{20}{2}} = \frac{20}{380}$$

$$P(G|J_2) = \frac{\binom{4}{2}}{\binom{18}{2}} = \frac{12}{306}$$

We also have $P(J_1) = P(J_2) = \frac{1}{2}$. We also have $P(G \cap J_i) = P(J_i)P(G|J_i)$ for $i = 1, 24$. Therefore

$$P(G) = \frac{1}{2}\left(\frac{20}{380} + \frac{12}{306}\right).$$

Definition 5.3. *Let A and B be events in a probability space (S, P). We say A and B are* ***independent*** *if*

$$P(A \cap B) = P(A)P(B).$$

Note that if $P(B) \neq 0$, then A and B are independent if and only if $P(A) = P(A|B)$. Non-independent events are said to be ***dependent****.*

Our solution to exercise 5.2 shows that the events G and J_1 are dependent.

Example 5.4 (Example 10.10 in [KT17]). *Consider the dicerolls as in Example 4.1. Let A be the event that the red die shows a 3 or a 5, and let B the event that you get doubles, i.e. the red and blue die show the same number. Then $P(A) = \frac{2}{6}$, $P(B) = \frac{6}{36}$, and we can directly compute that $P(A \cap B) = \frac{2}{36}$. So A and B are independent.*

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2