MAT344 Lecture 17

2019/July/16

1 Announcements

2 This week

This week, we are talking about

1. Recurrence equations

3 Recap

Last time we talked about

1. Ordinary and exponential generating functions

4 Recurrence Equations (Ch. 9 in [KT17])

We have already seen recurrences earlier in the semester where we used them to prove that certain objects are counted by the same formula. For example, we showed that Catalan numbers C_n count triangulations of a convex polygon by showing that they satisfy the same recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

and have the same initial conditions. Similarly, we argued that domino tilings of a $2 \times n$ rectangle are counted by Fibonacci numbers.

Last week, using generating functions, we were able to "solve" the recurrence equation

$$a_n = 3a_{n-1} - 1$$

and $a_0 = 2$. What do we mean by solving a recurrence equation? We were able to establish the formula

$$a_n = \frac{1}{2} + \frac{3}{2}3^n.$$

A formula for a sequence is always better than a recursion. Just think about the number of operations required to compute a_n using the recursive definition versus plugging in into the formula.

5 Linear Recurrence Equations (Ch. 9.2 in [KT17])

Consider the Fibonacci recurrence

$$F_n - F_{n-1} - F_{n-2} = 0.$$

this is a **linear recurrence**, because we can compute the n-th term as a linear combination of the previous terms.

Definition 5.1. *A* ***linear recurrence*** *is a recurrence equation of the form*

$$c_0 a_{n+k} + c_1 a_{n+k-1} + \ldots + c_k a_n = g(n).$$

Compare this to the Catalan recurrence, which is not linear. The RHS of the Fibonacci recurrence is zero, and we will refer to this as a **homogeneous** linear recurrence.

6 Advancement Operators (Ch. 9.3 in [KT17])

The theory of linear recurrence equations is very similar to that of linear differential equations.

Example 6.1 (Example 9.4. in [KT17])**.** *Solve the differential equation*

$$\frac{d}{dx} f = 3fm$$

with the initial condition $f(0) = 2$.

If you have seen differential equations, you know that the solution is $f(x) = 2e^{3x}$.

For differential equations, we apply the operator $\frac{d}{dx}$ (or its powers) to a (differentiable) function, and look for a solution.

For recurrence equations, we replace differentiable functions by sequences of numbers (or, functions $f : \mathbb{Z} \to \mathbb{R}$), and the operator $\frac{d}{dx}$ by the **advancement operator** A defined by

$$Af(n) = f(n+1).$$

For example, we could represent the Fibonacci sequence by the function

$$F(n) = F_n,$$

and then the recurrence, rewritten in terms of the advancement operator is

$$(A^2 - A - 1)f(n) = A^2 f(n) - Af(n) - f(n) = 0.$$

Before we solve this equation, let's take a look at an easier one.

Example 6.2 (Example 9.6 in [KT17])**.** *Suppose that the sequence* $\{s_n | n \geq 0\}$ *satisfies* $s_0 = 3$ *and* $s_{n+1} = 2s_n$ *for* $n \geq 1$. *Find an explicit formula for* s_n.

Solution: After some thought, we can guess that the solution is $s_n = 3 \cdot 2^n$, but let us rewrite it in terms of the advancement operator. We have

$$\begin{aligned} As(n) &= 2s(n) \\ As(n) - 2s(n) &= 0 \\ (A-2)s(n) &= 0 \end{aligned}$$

Notice that the advancement polynomial $(A - 2)$ has a root exactly at 2, and the solution is $s_n = 3 \cdot 2^n$.

Example 6.3 (Example 9.7 in [KT17])**.** *Find all solutions to the advancement operator equation*

$$(A^2 + A - 6)f(n) = 0.$$

Solution: We factor the polynomial $A^2 + A - 6 = (A + 3)(A - 2)$. If we write the equation now, we see that

$$(A+3)(A-2)f(n) = 0. \tag{1}$$

Note that any solution to $(A - 2)f(n) = 0$ or $(A + 3)f(n) = 0$ is a solution to equation (1).

For example, if $f(n) = c2^n$ for some constant c (as in the previous example), then $f(n)$ is still a solution. Also, by a similar logic, any function of the form $f(n) = c(-3)^n$ is also a solution. We will try to find all the solutions. Let $f(n) = c_1 2^n + c_2(-3)^n$, and apply $(A + 3)(A - 2)$. We have

$$\begin{aligned} (A+3)(A-2)f(n) &= (A+3)(c_1 2^{n+1} + c_2(-3)^{n+1} - 2(c_1 2^n + c_2(-3)^n)) \\ &= (A+3)(-5c_2(-3)^n) \\ &= 0. \end{aligned}$$

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2