MAT344 Lecture 16

2019/July/11

1 Announcements

2 This week

This week, we are talking about

1. The generalized binomial theorem

3 Recap

Last time we talked about

1. Encoding counting problems in power series

4 The generalized binomial theorem (Chapter 8.3 in [KT17])

Recall the following corollary of the Binomial Theorem:

Corollary 4.1. *If $p \geq 1$ is a positive integer, then*

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n.$$

Note that this means that the generating function for the number of n-subsets of a p-element set is $(1+x)^p$.

The right-hand side $(1+x)^p$ makes sense for p any real number, but binomial coefficients are only defined for $p, n \in \mathbb{Z}_{\geq 0}$.

Definition 4.2. *The* ***generalized binomial coefficient****, for $n \geq 0$, $p \in \mathbb{R}$ is*

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!},$$

note that $\binom{p}{0} = 1$.

Note that this definition agrees with our definition of binomial coefficients for $p \in \mathbb{Z}_{\geq 0}$.

Example 4.3.

$$\binom{-2}{5} = \frac{(-2)(-3)(-4)(-5)(-6)}{5!} = -6$$

It turns out that with this definition, the Binomial Theorem generalizes

Theorem 4.4 (Theorem 8.9. in [KT17])**.** *For all $p \in \mathbb{R}, p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n.$$

Why is Theorem 4.4 useful? If we have a generating function of the form $f(x) = (1 + ax)^p$, then the coefficient of x^n is $\binom{p}{n} a^p$ (for $p \in \mathbb{R}$).

Using the method of partial fractions, we can apply Theorem 4.4 to more functions.

Exercise 4.5 (Example 8.1. in [Mor17])**.** *Suppose we have a generating function*

$$f(x) = \frac{1+x}{(1-2x)(2+x)}$$

What is the coefficient of x^n?

Solution: We want to write

$$f(x) = \frac{1+x}{(1-2x)(2+x)} = \frac{A}{1-2x} + \frac{B}{2+x}.$$

for some $A, B \in \mathbb{R}$. Note that this leads to

$$\frac{A(2+x) + B(1-2x)}{(1-2x)(2+x)} = \frac{1+x}{(1-2x)(2+x)}.$$

and therefore

$$A(2+x) + B(1-2x) = 1 + x.$$

Since this is an equality between two polynomials, it should hold for specific values of x. If we let $x = -2$, we see that

$$5B = -1,$$

or, $B = \frac{-1}{5}$, and if we let $x = \frac{1}{2}$, we see that

$$\frac{5}{2}A = \frac{3}{2}$$

or, $A = \frac{3}{5}$. Therefore we have

$$f(x) = \frac{\frac{3}{5}}{1-2x} - \frac{\frac{1}{5}}{2+x}.$$

We can already use Theorem 4.4 on the first term, but the $(2+x)$ denominator is still a problem. So we rewrite

$$\frac{\frac{1}{5}}{2+x} = \frac{\frac{1}{10}}{1+\frac{x}{2}}.$$

Then we can apply Theorem 4.4 to both terms, and find that the coefficient of x^n in $f(x)$ is

$$\frac{3}{5}2^n - \frac{1}{10}(\frac{-1}{2})^n$$

Example 4.6 (Example 8.6 in [KT17])**.** *Find the number of integer solutions to the equation*

$$x_1 + x_2 + x_3 = n$$

with $x_1 \geq 0$ even, $x_2 \geq 0$ and $0 \leq x_3 \leq 2$.

This problem would be very difficult without using generating functions, as we don't have a good way to represent this computation as stars and bars.

Solution: The generating function for the problem is easy to write down, we just multiply the generating functions for the terms together to get

$$f(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \left(1 + x + x^2\right) = \frac{1+x+x^2}{(1-x^2)(1-x)}.$$

We again want to rewrite this as sums of simpler rational functions. Since the denominator has a double root, we want to solve

$$\frac{1+x+x^2}{(1-x^2)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

for A, B, C constants. We again clear the denominators to get

$$1 + x + x^2 = A(1-x)^2 + B(1-x^2) + C(1+x)$$

substituting $x = 1$ gives

$$3 = 2C,$$

or, $C = \frac{3}{2}$. substituting $x = -1$ gives

$$1 = 4A,$$

or, $A = \frac{1}{4}$. Then we can compare the x^2 terms on both sides to obtain

$$1 = A - B$$

and we get $B = -\frac{3}{4}$.

So we may write

$$f(x) = \frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}$$

so the coefficient of x^n in $f(x)$ is

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2}.$$

(note that this is an integer).

5 Generating functions and recursion

Example 5.1 (Example 8.6. in [Mor17]). *Consider the recursively defined sequence $a_0 = 2$, and for every $n \geq 1, a_n = 3a_{n-1} - 1$. Find an explicit formula for a_n in terms of n.*

Solution: The sequence a_n has generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$. How can we use the fact that $a_m = 3a_{m-1} - 1$? Note that

$$\begin{array}{rllllll} a(x) = a_0 & + a_1 x & + a_2 x^2 & + \dots & + a_m x^m & + \dots \\ -3xa(x) = & - 3a_0 x & - 3a_1 x^2 & - \dots & - 3a_{m-1} x^m & - \dots \\ a(x) - 3xa(x) = a_0 & - x & - x^2 & - \dots & - x^m & - \dots \end{array}$$

Therefore, we have the following equality

$$(1-3x)a(x) = 3 - (1 + x + x^2 + \dots)$$

and since $(1 + x + x^2 + \dots) = \frac{1}{1-x}$, we have the equality between generating functions

$$(1-3x)a(x) = 3 - \frac{1}{1-x},$$

or, equivalently,

$$a(x) = \frac{3}{1-3x} - \frac{1}{(1-x)(1-3x)}$$

we can use the method of partial fractions to rewrite this as

$$a(x) = \frac{\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}.$$

The coefficient of x^n is therefore

$$a_n = \frac{1}{2} + \frac{3}{2}3^n.$$

We can check the first few terms of this sequence, in particular,

$$\begin{aligned} a_0 &= \frac{1}{2} + \frac{3}{2} = 2 \\ a_1 &= \frac{1}{2} + \frac{3}{2}3 = 5 \\ a_2 &= \frac{1}{2} + \frac{3}{2}3^2 = 14 \end{aligned}$$

Which agrees with the recurrence relation $a_n = 3a_{n-1} - 1$.

6 Exponential generating functions

We declared that the function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

is the generating function for the sequence a_n (that probably represents the answer to a counting problem). These sort of generating functions are based on the function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We could have chosen another series to base the encoding of our counting problems on, for example, the Taylor series of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Definition 6.1 (Definition 9.6. in [Mor17])**.** *The* ***exponential generating function*** *of a sequence $a_0, a_1, \ldots$ is*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

So, the coefficients of an exponential generating function and an ordinary generating function differ by a factor of $n!$. We will also use exponential generating functions to encode answers to counting problems, and we'll see that they are useful when we consider objects where the order matters.

Example 6.2. *The exponential generating function for the number of binary strings of length n is*

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}.$$

compare this with the ordinary generating function for the same sequence

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

The difference between the two functions e^{2x} and $\frac{1}{1-2x}$ is that they have different algebraic properties, and this makes one or the other more convenient in certain situations. For example, the way exponential generating functions multiply is more convenient to use when the order matters. Why is this? Consider how exponential generating functions multiply. If $a(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$ and $b(x) = \sum_{l=0}^{\infty} b_l \frac{x^l}{l!}$, then

$$a(x)b(x) = \sum_{n=0}^{\infty} \sum_{k+l=n} a_k \frac{x^k}{k!} b_l \frac{x^l}{l!} \qquad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \frac{n!}{k!(n-k)!} \frac{x^n}{n!}$$

and we see the coefficient $\frac{n!}{k!l!} = \frac{n!}{k!(n-k)!}$ appear, which corresponds to the number of binary strings of length n with exactly k 0s.

For example, note that in the example above, we could represent a 0-string of length n (or a 1-string of length n) by the exponential generating function e^x or the ordinary generating function $\frac{1}{1-x}$. If we want to make a binary string of length n, we can just multiply the two exponential generating functions together

$$e^x e^x = e^{2x}$$

and get the correct answer. If we multiply the two ordinary generating functions together

$$\frac{1}{1-x}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

then we get the number of ways of picking a total n 0s and 1s, i.e. a $\{0,1\}$-combination with repetition. We could also interpret this as $\{0,1\}$-strings when anagrams are considered the same.

Example 6.3 (Example 8.16 in [KT17]). *Find the number of ternary strings in which the number of 0s is even.*

Solution: The exponential generating function for the 1s and 2s in the sequence is just e^x. To get an even number of 0s, we need the function

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}.$$

(Note that this is not the function e^{2x}). How can we represent this as a function? Note that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

and therefore

$$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \ldots$$

so the exponential generating function for the 0s is $\frac{e^x+e^{-x}}{2}$.

Then we multiply the exponential generating functions for the letters together to get

$$\frac{e^x + e^{-x}}{2} e^x e^x = \frac{e^{3x} + e^x}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right).$$

so the number of ternary strings with an even number of 0s is

$$\frac{3^n + 1}{2}.$$

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2, 5

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 2, 3, 4