MAT344 Lecture 15

2019/July/9

1 Announcements

2 This week

This week, we are talking about

1. Generating Functions

3 Recap

Last time we talked about

1. Inclusion-Exclusion

4 Generating Functions (Chapter 8 in [KT17])

Generating functions are a powerful tool used to answer enumerative problems. We will represent answers to counting problems as formal power series, and this will enable us to use algebra and calculus.

Exercise 4.1 (Example 8.4 in [KT17])**.** *Find the number of ways to distribute n apples to 5 children in a way that each child gets at least one apple.*

Solution: We already know that the answer to this question is $\binom{n-1}{4}$ (using stars and bars). We will find an alternative way of thinking about the problem.

Let's reinterpret the problem slightly. We want to give some number of apples to each of the 5 children. For each child, there is one way to give a child k apples, regardless of what k is. We will represent this as the **formal power series** (formal, because we are not concerned if the series is convergent)

$$\sum_{i=1}^{\infty} x^i = x^1 + x^2 + x^3 + \dots$$

How should we think of this formula? The formula represents all the possible outcomes of giving apples to the child. The + signs separating the terms of the series in the above formula can be thought of as "or"-s separating mutually exlusive statements. The term x^k means that this child gets k apples. The coefficient (in this case, one) of the term x^k means that there is one way of giving k apples to this child.

Since we have five children, we should distribute some number of apples to all of them. How can we represent this? The + signs represent "or" statements, and now we want to give some number of apples to the first child, *and* the second, *and* the third, and so on. The consistent way to do this is to **multiply** the series corresponding to giving apples to the different children. Let's do that and see what we get

$$(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)$$

What does this product represent? Each term of each of the factors represent one instance of giving some number of apples to a child. If we multiply out the series and collect the terms, the x^n term will represent a situation where we have distributed a total of n apples to the 5 children.

How do we multiply these series? Just like with polynomials, we multiply together one term from each of the factors. So, after combining terms, the coefficient of x^n will represent the number of ways of distributing a total of n apples among the children. What is the coefficient of x^n? We have to select a total of n "powers of x" from 5 factors, and we have to select at least one from each. So we see that there are $\binom{n-1}{4}$ ways of doing this.

In the above solution, we used the series as a tool, but our argument was still a combinatorial one.We could have argued that since we have

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$$

(for $|x| < 1$), we then have

$$\frac{x}{1-x} = x + x^2 + \ldots = \sum_{n=1}^{\infty} x^n.$$

We say that $\frac{x}{1-x}$ is the **generating function** associated to the counting problem of distributing n apples to 1 child.

We then have ,

$$(x^1 + x^2 + x^3 + \ldots)^5 = \left(\frac{x}{1-x}\right)^5 = \frac{x^5}{(1-x)^5}.$$

The generating function of the counting problem of distributing n apples to 5 children is then $\frac{x^5}{(1-x)^5}$.

If we can write $\frac{x^5}{(1-x)^5}$ as a power series, the coefficients should be the answer to out question. Notice that

$$\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = \frac{4!}{(1-x)^4}$$

therefore

$$\begin{aligned}
\frac{x^5}{(1-x)^5} &= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) \\
&= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(1 + x + x^2 + \ldots\right) && \text{we can differentiate a series term by term} \\
&= \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4} \\
&= \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3)}{4!} x^{n+1} \\
&= \sum_{n=0}^{\infty} \binom{n}{4} x^{n+1}
\end{aligned}$$

so again we find that the coefficient of x^n in this series is $\binom{n-1}{4}$.

The following easy proposition is useful for many enumerative problems

Proposition 4.2 (Proposition 7.13. in [Mor17])**.** *For any positive integer k,*

$$1 + x + x^2 + \ldots + x^k = \frac{1-x^{k+1}}{1-x}.$$

Example 4.3 (c.f. [Mor17] Example 7.14)**.** *You are playing a dice game, using regular 6-sided dice. In how many ways are there for you to roll a total of 12 on four dice?*

Solution: We can represent each die roll by the generating function

$$(x + x^2 + x^3 + x^4 + x^5 + x^6) = x\frac{1-x^6}{1-x}$$

and the total of the four dice rolls by the generating function

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^4 = x^4 \left(\frac{1 - x^6}{1 - x}\right)^4$$

We could multiply out the function and find the answer (the coefficient of x^{12}), but we can be a bit more clever by manipulating the function first.

First we can cancel out the x^4 factor, so we are looking for the coefficient of x^8 in

$$\left(\frac{1 - x^6}{1 - x}\right)^4$$

We may apply the binomial theorem to obtain

$$\begin{aligned}(1 - x^6)^4 &= \binom{4}{0}(-x^6)^0 + \binom{4}{1}(-x^6)^1 + \binom{4}{2}(-x^6)^2 + \binom{4}{3}(-x^6)^3 + \binom{4}{4}(-x^6)^4 \\ &= 1 - 4x^6 + 12x^{12} - 4x^{18} + x^{24} \qquad (1)\end{aligned}$$

Now also note that

$$(1 - x)^{-4} = \frac{1}{(1 - x)^4} = (1 + x + x^2 + \ldots)^4 \qquad (2)$$

and similarly to our first example today, the coefficient of the x^n term is $\binom{n+3}{3}$.

So, if we are looking for the coefficient of x^8 in $\left(\frac{1-x^6}{1-x}\right)^4$, the terms with powers x^{12} and higher in (1) will not contribute to this. The only way of getting x^8 is to either take the 1 term from (1) and the $\binom{11}{3}x^8$ term from () or to take the $-4x^6$ term from (1) and the $\binom{5}{3}x^2$ term from (4) for a total coefficient of

$$\binom{11}{3} - 4\binom{5}{3}$$

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 2