MAT344 Lecture 14

2019/July/4

1 Announcements

2 This week

This week, we are talking about

1. Inclusion-Exclusion

3 Recap

Last time we talked about

1. Prüfer codes

4 Inclusion-Exclusion (Chapter 7 in [KT17])

The first counting technique we discussed was the *addition principle*, which says that if A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|.$$

This is great, but not of much use if the two sets are not disjoint.

If A and B share elements (that is, if $A \cap B \neq \emptyset$), then $A + B$ is an overcount for $|A \cup B|$. We count every element of A once, and every element of B once. This results in us having counted elements both in A and B *twice*, so the exact count is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Notice how this formula generalizes the addition principle. Inclusion-Exclusion is the generalization of this simple observation to the case when more sets are involved.

Example 4.1. *What happens when we have 3 sets? Let* $A = \{1, 2, 3, 4\}, B = \{2, 4, 5\}, C = \{2, 5, 6\}$. *Then* $|A| = 4, |B| = |C| = 3$ *and*

$$A \cup B \cup C = \{1, 2, 3, 4, 5, 6\},$$

so $|A \cup B \cup C| = 6$. *The count* $|A| + |B| + |C| = 10$ *is an overcount. We counted* $A \cap B = \{2, 4\}$, $A \cap C = \{2\}$ *and* $B \cap C = \{2, 5\}$ *twice. If we subtract the sizes of these pairwise intersections, we get* $10 - 2 - 1 - 2 = 5$, *which is still not correct. So far we counted the triple intersection* $A \cap B \cap C = \{2\}$ *three times (as part of* A, B, *and* C*), then subtracted it three times (as part of* $(A \cap B), (A \cap C)$, *and* $(B \cap C)$. *So we still need to add it back once. Figure 1 shows a figure of a Venn diagram of the situation.*

This leads us to the following formula

Proposition 4.2 (Inclusion-Exclusion for three sets)**.** *Let* A, B, C *be sets, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

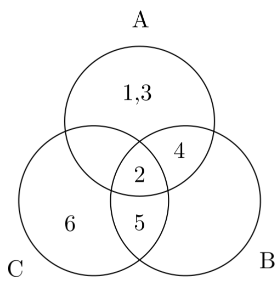
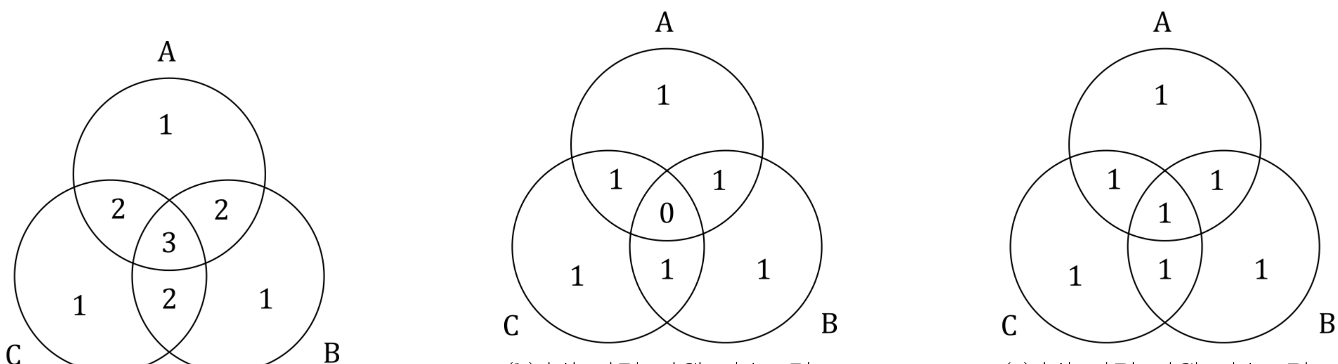


Figure 1: A Venn diagram



(a) $|A| + |B| + |C|$

(b) $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Figure 2: Inclusion-Exclusion on three sets

Proof. Let us draw a Venn diagram (see Figure 2) that illustrates how many times the elements in each intersection are counted. Notice how at each step, elements in one more "layer" of sets are counted correctly (with multiplicity one).

Q.E.D.

Exercise 4.3 (Example 2.1.1 in [Gui18])**.** *Find the number of solutions to the equation*

$$x_1 + x_2 + x_3 = 7$$

with $0 \le x_1 \le 2, 0 \le x_2 \le 4, 0 \le x_3 \le 3$.

Solution: Recall that if we didn't have the upper bounds on the variables, we could model this with a bars and stars computation. We have 7 stars and we want to separate them into 3 piles (so we need 2 bars). The number of solutions would be $\binom{7+3-1}{3-1} = \binom{9}{2}$. We can interpret this number as an overcount, and try to subtract the number of solutions which violate the conditions.

For example, how many of the $\binom{9}{2}$ solutions have $3 \le x_1$? We know how to turn this into a bars and stars problem, and the answer would be $\binom{4+3-1}{3-1} = \binom{6}{2}$. Similarly, $\binom{2+3-1}{3-1}$ of the solutions have $5 \le x_2$, and $\binom{3+3-1}{3-1}$ of the solutions have $4 \le x_2$. If we subtract these we get

$$\binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2}.$$

What does this number represent? We counted the solutions to $x_1 + x_2 + x_3 = 7$ in nonnegative integers, and subtracted one for each solution where $x_1 \ge 3, x_2 \ge 5$, or $x_3 \ge 4$. Do we have the correct count? We should be careful here, since for example we counted the solution $x_1 = 3, x_2 = 0, x_3 = 4$ once and subtracted it *twice*. First when we considered solutions with $x_1 \ge 3$ and a second time when we considered solutions with $x_3 \ge 4$. There is just one solution with $x_1 \ge 3$ and $x_3 \ge 4$, so we add this back. We should also consider other pairs of variables, but there are no solutions with $x_2 \geq 5$ and $x_1 \geq 3$, and similarly there are no solutions with $x_2 \geq 5$ and $x_3 \geq 4$, (or with all three conditions violated), so the total count is

$$\binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2} + 1$$

In general, it is useful to think of the above examples as a set X and a family $\mathcal{P} = \{P_1, \ldots, P_m\}$ of **properties**. What we mean by a property is that for every element $x \in X$ and every property P_i, x either satisfies property P_i or it does not. Inclusion-exclusion is about determining the number of elements that satisfy *none* of the properties. For example, in Exercise 4.3, the property P_1 was satisfied by a solution to $x_1 + x_2 + x_3 = 7$ if $x_1 \geq 3$.

Theorem 4.4 (Theorem 2.1.2 and Corollary 2.1.3 in [Gui18], (Principle of Inclusion-Exclusion))**.** *If $A_i \subseteq X$ for $1 \leq i \leq n$ then*

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\{i_1, i_2, \ldots, i_k\} \subseteq [n]} \left| \bigcap_{j=1}^{k} A_{i_j} \right|.$$

Let A_i^c denote the complement (in X) of A_i. Then an alternative way of stating the princuple of inclusion-exclusion is

$$\left| \bigcap_{i=1}^{n} A_i^c \right| = |X| + \sum_{k=1}^{n} (-1)^{k} \sum_{\{i_1, i_2, \ldots, i_k\}} \left| \bigcap_{j=1}^{k} A_{i_j} \right|.$$

Proof. We will prove the second formulation (the first one is equivalent). We need to show that every element of $\cap_{i=1}^{n} A_i^c$ is counted once by the right hand side and every other element of X is counted zero times. If $x \in \cap_{i=1}^{n} A_i^c$, then for every A_i, $x \notin A_i$, so x is in none of the sets involving A_i, and is counted exactly once by $|X|$.

If $x \notin A_i^c$, then on the RHS it is counted once by $|X|$, and it is counted for some of the values $i_1, i_2, \ldots, i_k$, $1 \leq m \leq k$, if x is not in the remaining sets A_j (for $j \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$. Then x is counted zero times by any term involving A_j with $j \notin \{i_1, i_2, \ldots, i_k\}$, either with a plus or minus sign, by each term involving only the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. Let's count in how many ways does this happen.

- There is the 1 term $|X|$, resulting in a count of $+1$
- There are k terms of the form $-|A_{i_m}|$, which results in a count of $-k$,
- There are $\binom{k}{2}$ terms of the form $|A_{i_l} \cap A_{i_m}|$, resulting in a count of $+\binom{k}{2}$
- In general, there are $\binom{k}{r}$ terms of the form $(-1)^r |A_{i_{s_1}} \cap A_{i_{s_2}} \cap \ldots A_{i_{s_r}}|$, resulting in a count of $(-1)^r \binom{k}{r}$.

Adding these up, we see that the number of times x is counted on the RHS is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i},$$

which we know equals zero.

Q.E.D.

Exercise 4.5 (Exercise 10.18. 8) in [Mor17])**.** *At a small university, there are 90 students that are taking either Calculus* or *Linear Algebra (or both). If the Calculus class has 70 students and the Linear Algebra class has 35 students, then how many students are taking both Calculus* and *linear algebra?*

Solution: Let A_1 be the set of students taking Calculus, and A_2 be the set of students taking linear algebra. The problem is asking for $|A_1 \cap A_2|$. From the problem statement, we know that

- $|A_1 \cup A_2| = 90$,
- $|A_1| = 70$,
- $|A_2| = 35$,

Theorem 4.4 tells us that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$

or, equivalently,

$$|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| = 70 + 35 - 90 = 15,$$

so, 15 students are taking both Calculus and Linear Algebra.

5 Counting surjections (Chapter 7.3 in [KT17])

Example 5.1 (Example 7.3. in [KT17]). *Let m and n be fixed positive integers and let X consist of all functions from $[n]$ to $[m]$. Then for each $i = 1, 2, \ldots, m$ and each function $f \in X$, we say that f is in A_i if there is no j such that $f(j) = i$. In other words, i is not in the image or output of the function f. For example, if $n = 5$ and $m = 3$, then the function f given by*

i	1	2	3	4	5
$f(i)$	2	3	2	2	3

We will use Theorem 4.4. Let X be the set of all functions from $[n]$ to $[m]$. For a subset $S \subseteq [m]$ of size k, we claim that

$$\left|\bigcap_{i \in S} A_i\right| = (m-k)^n.$$

This is true because a function f that is in $\bigcap_{i \in S} A_i$ is a string of length n from an alphabet consisting of $m - k$ letters.

Then by Theorem 4.4, the number $S(n, m)$ of surjections from $[n]$ to $[m]$ is

Theorem 5.2 (Theorem 7.8. in [KT17]).

$$S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

(you might recognize this formula from the Tutorials this week)

Exercise 5.3 (Chapter 7, Exercise 15 in [KT17]). *A teacher has 10 books (all different) that she wants to distribute to four students, ensuring that each of them gets at least one book. In how many ways can she do this?*

Note: You may think that we could answer this by first giving one book to each one of them (there are $10 \cdot 9 \cdot 8 \cdot 7$ ways of doing this, and then distributing the rest in 4^6 ways. But this is an overcount, and it is not so easy to see how much we overcounted each case by.

Solution: This is equivalent to counting surjections from [10] to [4], so the answer is

$$S(10, 4) = \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^{10} = 4^{10} - 4 \cdot 3^{10} + 6 \cdot 2^{10} - 4.$$

6 Counting derangements (Chapter 7.4 in [KT17])

Example 6.1 (Example 7.4. in [KT17]). *Let m be a fixed positive integer and let X consist of all bijections from $[m]$ to $[m]$. Note that these are in bijection with permutations of $[m]$. Let a permutation σ be in A_i if $\sigma(i) = i$. A permutation in $\bigcap A_i^c$ is a* ***derangement****. For example, the permutation σ is a derangement, while π is not*

i	1	2	3	4
$\sigma(i)$	2	4	1	3

i	1	2	3	4
$\pi(i)$	2	4	3	1

For a k-element subset $S \subseteq [n]$, we claim that

$$\left| \bigcap_{i \in S} A_i \right| = (n-k)!.$$

This is true because we have to keep $\sigma(i) = i$ for $i \in S$, and the remaining $(n-k)$ elements can be permuted arbitrarily. Similarly to 5.2, this leads to

Theorem 6.2 (Theorem 7.10. in [KT17])**.** *For each positive integer n, the number d_n of derangements of $[n]$ is*

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

Exercise 6.3 (Theorem 7.11. in [KT17])**.** *n men wearing top hats go to a ball. They check in their top hats with a Hat Check person. Later in the evening, the mischeivous hat check person decides to mix up the hats randomly. What is the probability that all n men receive a hat other than their own? Find the limit as $n \to \infty$.*

Solution: The hats can be redistributed in $n!$ ways, so we are looking for the number

$$\frac{d_n}{n!} = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!}{n!} \tag{1}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} \tag{2}$$

$$= (-1)^k \frac{1}{k!}. \tag{3}$$

and this is the Taylor series expansion of e^x evaluated at $x = -1$. So the answer is

$$\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}.$$

References

[Gui18] David Guichard. *Combinatorics and Graph Theory*. Open access, 2018. Available at https://www.whitman.edu/mathematics/cgt_online/book/. 2, 3

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 4, 5

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 3