MAT344 Lecture 12

2019/June/13

1 Announcements

2 This week

This week, we are talking about

1. Planar graphs
2. Labeled trees

3 Recap

Last time we talked about

1. Planar graphs

4 The Euler characteristic

If we want to identify which graphs are planar, we have to look for something that involves the planar drawing. Any drawing of a graph has vertices and edges, but a planar drawing also has

Definition 4.1. *Given a planar drawing of a graph, a* ***face*** *of the drawing is a region of the plane bounded by vertices and edges not containing any other vertices and edges.*

What patterns do we notice between the vertices, edges and faces of a planar graph? Let's compute some examples.

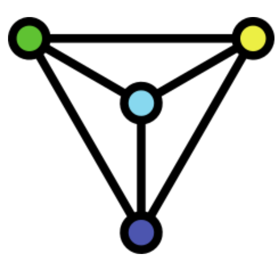

Figure 1: A planar graph

Example 4.2.

For example, the graph on Figure 1 has 4 vertices, 6 edges, and 4 faces (one is the unbouded "outside" face).

The triangle graph K_3 has 3 vertices, 3 edges and 2 faces.

The path graphs P_n have n vertices, $n-1$ edges and one face.

The cycle graphs C_n have n vertices, n edges and 2 faces.

Theorem 4.3 (Euler, Theorem 5.11 in [KT17])**.** *For any planar drawing of a connected graph G, let v, e, f denote the number of vertices, edges and faces in the drawing. Then*

$$v - e + f = 2.$$

To prove the theorem, we need some basic results about graphs:

Theorem 4.4 (Theorem 12.27 in [Mor17])**.** *The following are equivalent for a graph T with n vertices:*

1. *T is a tree.*
2. *T is connected and has $n - 1$ edges.*
3. *T has no cycles and has $n - 1$ edges.*
4. *T is connected, but deleting any edge leaves a disconnected graph.*

Proof of Theorem 4.3. Let $V(G), E(G)$, and $F(G)$ denote the set of vertices, edges and faces of G. Let $v = |V(G)|, e = |E(G)|, f = |F(G)|$. Since the graph is connected, if it has v vertices, it has at least $v - 1$ edges (by Theorem 4.4). We will use induction on $e - v$, with the base case being $e - v = 1$. In this case, G is a tree (using Theorem 4.4 again), and therefore contains no cycles. Therefore the number of regions in the planar drawing is 1, and in this case, we have

$$v - e + f = v - (v - 1) + 1 = 2,$$

so the base case holds.

Now assume for induction that any connected planar graph with $e - v < n$ satisfies $v - e + f = 2$.

For the inductive step, suppose that $e - v = n$, with $n \geq 2$. Then G must have a cycle (using Theorem 4.4 again). Remove one edge from one of the cycles in G. Call the resulting connected graph G'. Then $|V(G')| = |V(G)| = v$, $|E(G')| = |E(G)| - 1 = e - 1$. We want to show that $|F(G')| = |F(G)| - 1 = f - 1$. Since the edge we removed was part of a cycle, it used to separate two regions in the drawing, and now those two regions are joined into one, and the other regions are unchanged, therefore $F(G') = f - 1$. So we have $|E(G')| - |V(G')| = e - 1 - v < n$, so applying the inductive hypothesis for the graph G', we see that

$$\begin{aligned} 2 &= |V(G')| - |E(G')| + |F(G')| \\ &= v - (e - 1) + (f - 1) \\ &= v - e + f \end{aligned}$$

Q.E.D.

Corollary 4.5. *Let G be a connected planar graph. Then every planar drawing of G has the same number of faces.*

Proof. $|V(G)|$ and $|E(G)|$ are determined independently of the drawing, and $|F(G)| = 2 - |V(G)| + |E(G)|$.

Q.E.D.

Corollary 4.6 (Theorem 5.12. in [KT17])**.** *If G is a connected planar graph and $|V| \geq 3$, then*

$$|E| \leq 3|V| - 6.$$

If, in addition, G has no cycles of length less than 4, then

$$|E| \leq 2|V| - 4$$

Proof. We will give a combinatorial proof, counting the number of edges we encounter as we move around the boundary of each face.

First, note that every edges is adjacent to either one or two faces. If it is adjacent to two faces, it separates them, and as we move around both of those faces, we will count the edge once for each face. If an edge is adjacent to just one face, it will still be counted twice as we move around the boundary of the face (once we move toward the "inside" of the face and once when we move "putward"). So every edge is counted exactly twice, so our count is $2|E|$.

Secondly, we look at all the faces and count how many edges surround that face. There must be at least 3 edges around each face, unless there is just one face (in this case G is a tree, so $|E| = |V| - 1$ and since $|V| \geq 3$, we are done). Therefore, we have counted at least $3|F|$ edges (there is some overcount here, but that's okay, we are looking for an inequality).

From the above, we learn that $2|E| \geq 3|F|$, or, equivalently

$$|F| \leq \frac{2|E|}{3}.$$

Using Euler's formula, $|F| = 2 - |V| + |E|$, we get

$$2 \leq |V| - |E| + \frac{2|E|}{3},$$

and some algebra yields

$$|E| \leq 3|V| - 6.$$

For the second part, we assume that G has no cycles of length less than 4. In this case, every face must be surrounded by at least 4 edges, and the rest of the argument is unchanged.

So we get the estimate $2|E| \geq 4|F|$, or

$$|F| \leq \frac{|E|}{2}.$$

And applying Euler's formula yields

$$2 - |V| + |E| \leq \frac{|E|}{2}$$

from which we get

$$|E| \leq 2|V| - 4.$$

Q.E.D.

Consider the case of $K_{3,3}$. It has 6 vertices and 9 edges. It also has no cycles of length 3 (or any odd-length cycles, since it's bipartite). Any planar graph that has 6 vertices and contains no 3-cycles must have at most

$$2 \cdot 6 - 4 = 8$$

edges. So $K_{3,3}$ is not planar.

5 Homeomorphic graphs and Kuratowski's Theorem

It turns out that K_5 and $K_{3,3}$ are the two graphs that determine planarity. Let us say what this means. We already know that any graph containing either a copy of K_5 or $K_{3,3}$ as a subgraph can not be planar. Consider the graph on Figure 2.

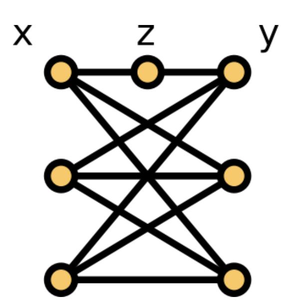

Figure 2: A graph homeomorphic to $K_{3,3}$

The first thing that we notice is that the graph looks very similar to $K_{3,3}$, except that we subdivided an edge between the top two vertices by adding an extra vertex z. In particular, the vertices x and y are no longer adjacent.

Definition 5.1. *If G is a graph and xy is an edge, then we can form a new graph G' called an **elementary subdivision** of G by adding a new vertex z and replacing the edge xy by edges xz and zy.*

As a result, the graph no longer contains a copy of $K_{3,3}$. The only possible subgraph that could be a copy of $K_{3,3}$ would be all vertices except z (for degree reasons), but as x is not adjacent to y, we can conclude that this graph does not contain $K_{3,3}$.

Should this affect planarity? As we are drawing the edges in the plane, all we care about are edge crossings. Having the vertex z in our graph adjacent to both x and y does not change the drawing problem in any meaningful way (note that we can not cheat by making an edge "pass through" z, since z is not adjacent to any vertex other than x and y).

Definition 5.2. *Two graphs G_1 and G_2 are said to be **homeomorphic** if they can be obtained from some graph G by a sequence of elementary subdivisions.*

Exercise 5.3. *If you know some topology, prove that graphs that are homeomorphic as graphs are also homeomorphic as topological spaces (with the subspace topology from $\mathbb{R}^2$.*

Our argument above can be generalized to the following proposition

Proposition 5.4. *Any graph containing a subgraph homeomorphic to K_5 or $K_{3,3}$ is nonplanar.*

Exercise 5.5. *Show that the Petersen graph (from Lecture 10) is not planar by finding a subgraph homeomorphic to $K_{3,3}$.*

What is much more surprising is that the converse of Proposition 5.4 is also true:

Theorem 5.6 (Kuratowski, 1930, Theorem 5.13 in [KT17])**.** *A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.*

In this week's tutorials you saw two operations on graphs, **edge deletion** and **edge contraction**.

Definition 5.7 (Definition 15.19 in [Mor17])**.** *Let G be a graph. Then H is a **minor** of G if we can construct H from G by a sequence of edge deletions, edge contractions and vertex deletions.*

Theorem 5.8 (Wagner, 1937, Theorem 15.20 in [Mor17])**.** *A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

6 Prüfer codes and Counting labeled trees

So far all the vertices of the graphs we considered were indistinguishable. In many practical problems where graphs are used (such as connecting cities by highways, for example), the vertices are distinguishable. How do we model this phenomenon with graphs? We will assign a **label** to each vertex, and we will only consider two graphs isomorphic if the labels also match, for example, the graphs on figure 3 are isomorphic *as graphs* but **not** isomorphic as *labeled graphs.*

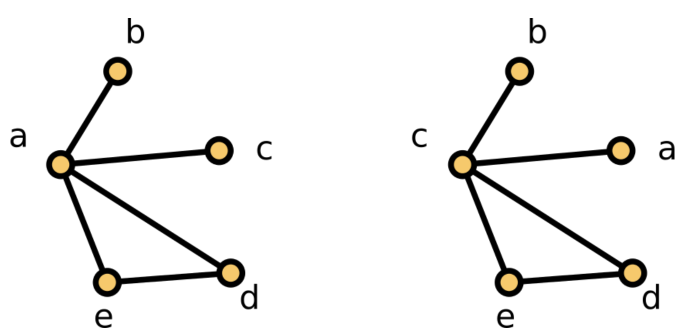

Figure 3: Two nonisomorphic labeled graphs

We will be focusing on labeled trees, and trying to answer the question: How many trees are there with the vertex set $[n]$?

For $n = 1$ or $n = 2$, there is just one labeled tree. For $n = 3$, there is still only one tree (the path graph P_3 on 3 vertices), but now it matters which of the 3 labels is assigned to the vertex of degree 2. So there are 3 labeled trees on 3 vertices.

Exercise 6.1. *Count all the labeled trees on 4 vertices.* ***Hint:*** *there are only two trees, the complete bipartite graph* $K_{1,3}$ *and the path graph* P_4

Theorem 6.2 (Cayley's formula, Theorem 5.15 in [KT17])**.** *The number* T_n *of labeled trees on* n *vertices is* n^{n-2}.

We will use an algorithm that systematically deconstructs the labeled tree T on n vertices, until it is the (unique) labeled tree on 2 vertices, and produces a string with $n - 2$ elements from the alphabet $[n]$. This string will be called the **Prüfer code** of the labeled tree, denoted Prüfer(T). If v is a leaf (a vertex of degree 1) of T, we will denote $T - v$ the labeled tree obtained from T by removing v (and the edge adjacent to it).

We define Prüfer(T) as follows

1. If T is the unique labeled tree on 2 vertices, return the empty string.
2. Else, let v be the leaf of T with the smallest label, and let u be its unique neighbor. Let i be the label of u. Return $(i, \text{Prüfer}(T - v))$.

Let's do an example, consider the labeled tree T on figure 4.

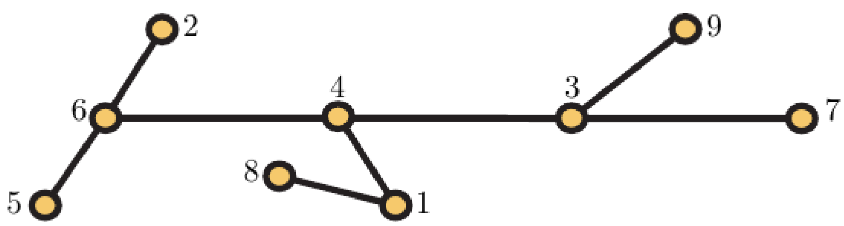

Figure 4: The graph T

Let v be the vertex with label 2 (it is the leaf with the smallest label). It is adjacent to 6, so

$$\text{Prüfer}(T) = (6, \text{Prüfer}(T - v).$$

Exercise 6.3. *Find the Prüfer code of* T.

Solution:

- The next smallest leaf is labeled 5, also adjacent to 6, so so far our code is 66.
- The next smallest leaf is labeled 6 (note that this just became a leaf), it is adjacent to 4, so our code so far is 664.
- Then 7 is deleted, and our code is 6643.
- Then 8 is deleted, our code is 66431.
- Then 1 is deleted, our code is 664314.
- Then 4 is deleted, our code is 6643143, and the remaining tree has 2 vertices.

So

$$\text{Prüfer}(T) = 6643143.$$

What does this accomplish? To a labeled n-vertex tree, we assigned an $[n]$-string of length $n - 2$. There are n^{n-2} many such strings, so if we can prove that this assignment is a bijection, we will have proved Theorem 6.2.

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 2, 4, 5

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 2, 4