MAT344 Lecture 11

2019/June/11

1 Announcements

Details about the midterm in the quercus announcement.

2 This week

This week, we are talking about

1. Planar graphs
2. Labeled trees

3 Recap

Last time we talked about

1. Euler walks and Hamilton paths
2. Graph coloring

4 Planar graphs (Chapter 5.5 in [KT17])

Given a map of certain countries, how many colors do we need to color the map if no two adjacent countries can have the same color? If you try with a couple of maps, you'll notice that four colors seem to be enough. It is also easy to draw a map where 4 colors are necessary, for example, Figure 1

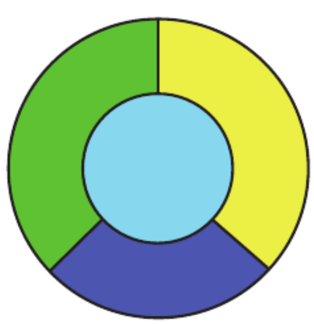

Figure 1: A map that requires four colors

We can easily translate this question into a graph coloring problem by constructing a graph where the vertices represent the countries and there is an edge between two vertices if the two countries share a border, see Figure 2.

How can four colors be enough? Last week we saw that there are graphs with arbitrarily large chromatic number. There must be something special about graphs that we get this way from maps of countries.

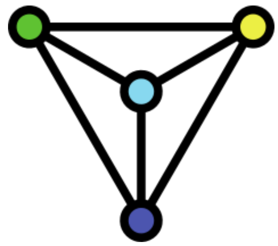

Figure 2: The graph representing the map in Figure 1

Definition 4.1. *A graph if* ***planar*** *if it can be drawn in the plane* ($\mathbb{R}^2$) *without edges crossing.*

Theorem 4.2 (Four color Theorem, Theorem 5.14 in [KT17])**.** *Every planar graph has chromatic number at most four.*

The proof of Theorem 4.2 was a gigantic effort, only finished in 1976, the published paper was pretty much unreadable and it relied on a computer checking many cases, and it contained several flaws (that were later fixed).

Definition 4.1 should give you some discomfort. We defined graphs abstractly, and noticed that it is not so easy to tell when two drawings represent the same graph. Especially if a graph is given to us as a drawing, it may be difficult to say if it is planar or not. It may be possible that this particular drawing has intersecting edges, but if we position the vertices differently we may be able to avoid this.

It also makes it very difficult to prove that a graph *isn't* planar, so we would like to have alternative characterizations of planar graphs.

We already know that certain graphs can't be planar, as Theorem 4.2 implies that if G is planar, then $\chi(G) \leq 4$. So any graph with a chromatic number at least 5 can not be planar. This leads to the following theorem.

Theorem 4.3. *Any graph that contains a copy of* K_5 *can not be planar.*

Could this be the characterization? Consider the complete bipartite graph $K_{3,3}$ on $3+3$ vertices, shown on Figure 3.

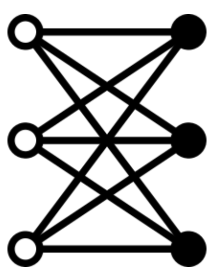

Figure 3: $K_{3,3}$

This graph clearly has $\chi(K_{3,3}) = 2$, but this way of drawing it has a lot of intersections. If we draw the graph slightly differently, for example, as in 4, it does not look that far from being planar.

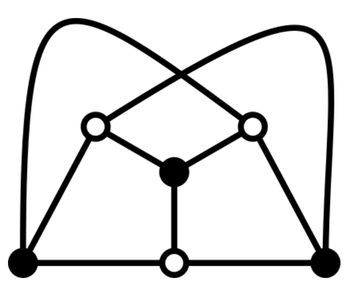

Figure 4: $K_{3,3}$

Exercise 4.4. *Let* $\{v_1, v_2, v_3\}$ *and* $\{u_1, u_2, u_3\}$ *be the vertices of the complete bipartite graph* $K_{3,3}$*. Consider the 4-cycle* u_1, v_1, u_2, v_2*. This 4-cycle divides the plane into two regions. The remaining two vertices* u_3 *and* v_3 *must both lie either inside or outside the 4-cycle, because they are adjacent. In both cases, find a contradiction, thereby showing that* $K_{3,3}$ *is not planar.*

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2