Tutorial 6

Partitions of $[n]$

Learning Objectives

> In this tutorial you will be determining the number of partitions of a set of distinct elements.
>
> These problems relate to the following course learning objectives: *Analyze a counting problem by proving an exact or approximate enumeration, or a method to compute one efficiently*, and *describe solutions to iterated processes by relating recurrences to induction and combinatorial identities.*

Counting Partitions

A *partition of* $[n]$ *into* k *parts* is a collection of k disjoint non-empty sets whose union is $[n]$. For example, $\{\{1,3\},\{2\},\{4\}\}$ is a partition of $[4]$ into 3 parts. To simplify notation, we would write this partition as $\{13,2,4\}$.

The number of partitions of $[n]$ into k sets is denoted $\left\{ {n \atop k} \right\}$. The partitions of $[4]$ into 3 parts are:

$$\{12,3,4\},\{13,2,4\},\{14,2,3\},\{1,23,4\},\{1,24,3\},\{1,2,34\}.$$

Hence $\left\{ {4 \atop 3} \right\} = 6$.

1. Determine $\left\{ {3 \atop 2} \right\}$, $\left\{ {4 \atop 2} \right\}$, and $\left\{ {5 \atop 4} \right\}$.
2. Show that $\left\{ {n \atop 1} \right\} = 1$ and $\left\{ {n \atop n} \right\} = 1$.
3. Show that $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

Formulas and Recursions

In any partition, the element 1 is either in a part by itself, or in a part with at least one other element.

4. Using this observation, give a bijective proof that

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}.$$

5. Compute $\left\{ {5 \atop 2} \right\}$ and $\left\{ {5 \atop 3} \right\}$.
6. Conjecture and prove a formula for $\left\{ {n \atop 2} \right\}$, either by giving a bijective proof, or using the recurrence and induction.

Let $S(n,k)$ denote the number of surjections from $[n]$ to $[k]$. Recall that we gave a method for counting these by inclusion-exclusion.

7. Show that $\left\{ {n \atop k} \right\} = \frac{S(n,k)}{k!}$.
8. Compute $\left\{ {100 \atop 3} \right\}$. How many iterations would this computation take when using the recurrence relation?

1. We have $\left\{ {3 \atop 2} \right\} = \binom{3}{2} = 3$, by considering which two elements go together;

   $\left\{ {4 \atop 2} \right\} = 4 + 3 = 7$, either we have a partition with one element alone 4 ways, or we have two parts of size 2, $\binom{4}{2}/2$ ways, dividing by two because choosing two elements gives the same partition as choosing their complement; and

   $\left\{ {5 \atop 4} \right\} = \binom{5}{2} = 10$, by choosing which elements go in the part of size 2.

2. There is only one way to put all elements into one part, and only one way to put them each into their own part.

3. Using $n - 1$ parts will mean having one part of size 2 and the rest of size 1. To do this, we choose two elements for the part of size 2, and the order they are chosen doesn't matter.

4. If 1 is in a part by itself, we can remove that part and partition the remaining $n - 1$ elements into $k - 1$ parts, $\left\{ {n-1 \atop k-1} \right\}$ ways. If it is not, then we first partition the remaining elements into k parts, $\left\{ {n-1 \atop k} \right\}$ ways, then place 1 into one of the k parts. These are disjoint options and cover all possibilities.

5. Using the recurrence and previous values, we have

$$\left\{ {5 \atop 2} \right\} = \left\{ {4 \atop 1} \right\} + 2\left\{ {4 \atop 2} \right\} = 1 + 14 = 15$$

   and

$$\left\{ {5 \atop 3} \right\} = \left\{ {4 \atop 2} \right\} + 3\left\{ {4 \atop 3} \right\} = 7 + 18 = 25.$$

6. $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$. This can be shown by induction, or by counting subsets of $[n]$ containing 1 as the first part of the partition. The other part will be the remaining elements, so we need to subtract the number of ways it can be empty.

7. A surjection gives us a labelled partition: each element of $[n]$ is placed into a set corresponding to an element of $[k]$, and no set is empty. To remove the labels, we divide by the number of ways to label these sets.

8. Since the number of surjections $[n]$ to $[3]$ is $3^n - 3 \cdot 2^n + 3$, we have $\left\{ {100 \atop 3} \right\} = (3^{99} - 2^{100} + 1)/2$. Reducing this by using the recurrence would take 95 iterations to get down to $\left\{ {5 \atop 3} \right\}$, the largest value of $\left\{ {n \atop 3} \right\}$ which we knew.