Learning Objectives

> In this tutorial you will be constructing and comparing proofs by algebra or induction with bijective arguments.
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> These problems relate to the following course learning objectives: *Describe solutions to iterated processes by relating recurrences to induction and combinatorial identities*, and *prove combinatorial identities by counting a set of objects in two ways.*

1 Identities

Prove each of the following in two ways: by algebraic manipulations or induction, and by a bijective argument, showing that each side counts the same set.

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
3. $2^n = \sum_{k=0}^{n} \binom{n}{k}$.
4. $\binom{n}{2} = \sum_{i=1}^{n-1} i$ for $n \geq 2$.
5. $\binom{2n+2}{3} = \sum_{i=1}^{n} (2i)^2$.

The Hemachandra numbers H_n count the number of ways to cover a length n strip with blocks of length 1 and 2. We have $H_1 = 1$, $H_2 = 2$, and we can show that $H_{n+1} = H_n + H_{n-1}$. The Fibonacci numbers have the same values, but shifted by one position, and Hemachandra predates Fibonacci by decades. Give bijective proofs of the following:

6. $H_{2n} = H_n^2 + H_{n-1}^2$ for $n \geq 1$.
7. $H_{2n+1} = H_n H_{n+1} + H_n H_{n-1}$ for $n \geq 1$.
8. Generalize to show $H_{k+n} = H_k H_n + H_{k-1} H_{n-1}$ for $k, n \geq 1$.

2 Catalan descriptions

Recall that the Catalan numbers C_n count the number of lattice paths from $(0, 0)$ to (n, n) which never contain a point (x, y) where $y > x$. They are given by $C_n = \binom{2n}{n}/(n + 1)$.

9. Show that C_n counts the number of ways to arrange n pairs of brackets.
10. Show that C_n counts the number of ways to arrange n identical coins into n boxes labelled $1, 2, \ldots, n$, so that boxes $1, \ldots, k$ contain at most k coins in total, for every $k \geq 1$.

1. Algebraically these are identical by definition. $\binom{n}{k}$ counts the number of subsets of size k from a set of size n, which can also be counted by asking which elements are not in the subset, given by $\binom{n}{n-k}$.

2. A small amount of algebra gives the identity. It can also be proven by counting subsets of size k from a set of size $n+1$. Consider one specific element. Each subset either contains the specific element, in which case there are $\binom{n}{k-1}$ ways to choose the remainder, or it doesn't contain it, in which case there are $\binom{n}{k}$. These are disjoint, so their sum is the total number of subsets of size k.

3. The induction begins with a base case of $n = 0$, and the inductive step is proven by using the identity from question 2, along with $\binom{n+1}{n+1} = \binom{n}{n}$ and $\binom{n+1}{0} = \binom{n}{0}$. For a bijective proof, the left side counts all possible subsets of a set of size n, by choosing one of two options (in the subset or not) for each element independently. The right side also counts the number of subsets, separated into cases by the size k of the subset, since there are $\binom{n}{k}$ subsets of size k, and each possible values of k is considered.

4. The induction begins with a base case of $n = 0$, and the inductive step is proven by using the identity from question 2 with $k = 2$. The left side counts the number of binary strings of length n containing exactly 2 ones, while the right side counts the same strings by selecting the first one, then counting how many possibilities are available for the second.

5. The induction is straightforward. The left side counts the number of binary strings of length $2n+2$ containing exactly 3 ones. Such a string can be separated $n+1$ pairs, and counted according to the first pair that contains a one. Suppose there are i pairs to the right of this one. Then $1 \leq i \leq n$, and there are either 2 ones in this pair, or not. If there are 2, then there are $2i$ choices for the remaining one. If not, then there are 2 positions within the pair for the one, and $\binom{2i}{2}$ choices for the remaining ones. Some algebra will show that $2i + 2\binom{2i}{2} = (2i)^2$.

6. H_{2n} counts the number of coverings of length $2n$. Every such covering can either be broken into two strips of length n, or contains a block of length 2 in the middle. The second case divides the strip into two strips of length $n-1$. These simpler coverings are counted by $H_n \cdot H_n$ and $H_{n-1} \cdot H_{n-1}$, since each part can be covered independently, and these can be added since the cases are disjoint.

7. A similar argument works for the odd indices. In order to prove these statements by induction, both of them must be used simultaneously, since the recursive formula involves both even and odd indexed terms.

8. Each covering of a strip of length $k+n$ can either be split into a strip of length k followed by a strip of length n, or contains a block of size 2 at that position, which breaks the strip into one of length $k-1$ and one of length $n-1$.

9. We can construct a bijective between paths and arrangements of brackets by writing (for each step right, and) for each step up. Since each path never crosses the line $y = x$, we will never have more) than (, so our bracketing will make sense. Similarly, given a string of brackets with n pairs of (and), we can construct a path which does not cross $y = x$. Hence, every path gives a bracketing, and every bracketing gives a path, so there are the same number of each.

10. There is a correspondence between coins in boxes and paths by considering each vertical line $x = k$ to correspond to box k, and the number of vertical steps at $x = k$ to correspond to the number of coins. Each path never crosses the line $y = x$, so the boxes $1, \ldots, k$ will contain at most k coins, and the total number of vertical steps will be n, so the number of coins in all boxes is n.