# MAT344 Problem Set 8
# (due Thursday, Nov/21, noon)

**Notes:**

- For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.
- When submitting to Crowdmark, please ensure that your uploads are *legible*, *correctly rotated*, and *properly matched with the correct problems.* Any improperly uploaded problem scans will not be graded.
- Any assignments submitted after the deadline will not be accepted.

## Part A

Three randomly chosen questions from this part will be marked.

**Problem 1.** For each of the infinite sequences below, give its generating function in closed form (i.e., not as an infinite sum). For example, for the sequence $(1, 1, 1, 1, \ldots)$, the generating function is $\frac{1}{1-x}$.

1. $[2^8, 2^7\binom{8}{1}, 2^6\binom{8}{2}, \ldots, \binom{8}{8}, 0, 0, 0, \ldots]$.
2. $[0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, \ldots]$.

**Problem 2.** Use a computer algebra system (for example, SageMath following example 8.6 in the textbook, Python, Mathematica, MATLAB) to find the coefficient of $x^{500}$ for the following generating functions:

1. the number of ways of making change in Canada,

$$\frac{1}{(1-x^5)(1-x^{10})(1-x^{25})(1-x^{100})(1-x^{200})}.$$

2. the number of ways of buying chicken McNuggets in Canada,

$$\frac{1}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{20})}.$$

**Problem 3.** Find the generating function for the number of non-negative integer solutions to

$$3x + 2y + 7z = n$$

**Problem 4.** Find the generating function for the number of ways to select $n$ balloons from white, gold, and blue balloons containing at least one white balloon, at least one gold balloon, and at most two blue balloons. How many ways are there to select 10 balloons subject to these requirements?

# Part B

Two randomly chosen questions from this part will be marked.

**Problem 5.** For the recursively-defined sequence $c_0 = 2, c_1 = 0$, $c_n = c_{n-1} + 2c_{n-2}$ for $n \geq 2$, use the method of generating functions to find an explicit formula for the $n$th term of the sequence.

**Problem 6.**

(a) Find the generating function for the number of ways of painting some number of identical balls red, yellow and blue, if at most 2 can be painted red, an even number must be painted yellow, and any number can be painted blue.

(b) Use the generating function you found to answer the question for $n$ balls.

**Problem 7.** You are analyzing a lottery game where players choose either to write a ternery string of length $n$, or choose two numbers from $[n]$. (The story for the player is to guess a sequence of $\triangle, \square, \bigcirc$, with a reward proportional to the number guessed correctly, or to choose two positions in the sequence where a $\bigcirc$ appears, with a reward inversely proportional to the number of $\bigcirc$ that appear. You don't need to be concerned with the story for this question.) The lottery commission would like to know how many player choices there are for different values of $n$, which you immediately recognize is $3^n + \binom{n}{2}$. Unfortunately, the lottery commissioner had heard of the hot new trends in combinatorics, and will only accept a generating function as a submission. Find and justify a generating function for this sequence.

# Part C

This question will be marked for completion only.

**Problem 8.** Construct a counting problem that can be solved using generating functions, but is difficult to solve using counting techniques you've learned earlier.