MAT344 Week 9

2019/Nov/11

1 This week

This week, we are talking about

1. Generating Functions

2 Recap

Before the break we talked about

1. Inclusion-Exclusion

3 Generating Functions (Chapter 8 in [KT17])

Generating functions are a powerful tool used to answer enumerative problems. We will represent answers to counting problems as formal power series, and this will enable us to use algebra and calculus.

Exercise 3.1 (Example 8.4 in [KT17])**.** *Find the number of ways to distribute n apples to 5 children in a way that each child gets at least one apple.*

Solution: We already know that the answer to this question is $\binom{n-1}{4}$ (using stars and bars). We will find an alternative way of thinking about the problem.

Let's reinterpret the problem slightly. We want to give some number of apples to each of the 5 children. For each child, there is one way to give a child k apples, regardless of what k is. We will represent this as the **formal power series** (formal, because we are not concerned if the series is convergent)

$$\sum_{i=1}^{\infty} x^i = x^1 + x^2 + x^3 + \dots$$

How should we think of this formula? The formula represents all the possible outcomes of giving apples to the child. The + signs separating the terms of the series in the above formula can be thought of as "or"-s separating mutually exlusive statements. The term x^k means that this child gets k apples. The coefficient (in this case, one) of the term x^k means that there is one way of giving k apples to this child.

Since we have five children, we should distribute some number of apples to all of them. How can we represent this? The + signs represent "or" statements, and now we want to give some number of apples to the first child, *and* the second, *and* the third, and so on. The consistent way to do this is to **multiply** the series corresponding to giving apples to the different children. Let's do that and see what we get

$$(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)(x^1 + x^2 + x^3 + \dots)$$

What does this product represent? Each term of each of the factors represent one instance of giving some number of apples to a child. If we multiply out the series and collect the terms, the x^n term will represent a situation where we have distributed a total of n apples to the 5 children.

How do we multiply these series? Just like with polynomials, we multiply together one term from each of the factors. So, after combining terms, the coefficient of x^n will represent the number of ways of distributing a total of

n apples among the children. What is the coefficient of x^n? We have to select a total of n "powers of x" from 5 factors, and we have to select at least one from each. So we see that there are $\binom{n-1}{4}$ ways of doing this.

In the above solution, we used the series as a tool, but our argument was still a combinatorial one.We could have argued that since we have

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$$

(for $|x| < 1$), we then have

$$\frac{x}{1-x} = x + x^2 + \ldots = \sum_{n=1}^{\infty} x^n.$$

We say that $\frac{x}{1-x}$ is the **generating function** associated to the counting problem of distributing n apples to 1 child.

We then have ,

$$(x^1 + x^2 + x^3 + \ldots)^5 = \left(\frac{x}{1-x}\right)^5 = \frac{x^5}{(1-x)^5}.$$

The generating function of the counting problem of distributing n apples to 5 children is then $\frac{x^5}{(1-x)^5}$.

If we can write $\frac{x^5}{(1-x)^5}$ as a power series, the coefficients should be the answer to out question. Notice that

$$\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = \frac{4!}{(1-x)^5}$$

therefore

$$\begin{aligned}
\frac{x^5}{(1-x)^5} &= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) \\
&= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(1 + x + x^2 + \ldots\right) && \text{we can differentiate a series term by term} \\
&= \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4} \\
&= \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3)}{4!} x^{n+1} \\
&= \sum_{n=0}^{\infty} \binom{n}{4} x^{n+1}
\end{aligned}$$

so again we find that the coefficient of x^n in this series is $\binom{n-1}{4}$.

The following easy proposition is useful for many enumerative problems

Proposition 3.2 (Proposition 7.13. in [Mor17])**.** *For any positive integer k,*

$$1 + x + x^2 + \ldots + x^k = \frac{1 - x^{k+1}}{1-x}.$$

Example 3.3 (c.f. [Mor17] Example 7.14)**.** *You are playing a dice game, using regular 6-sided dice. In how many ways can you to roll a total of 12 on four dice?*

Solution: We can represent each die roll by the generating function

$$(x + x^2 + x^3 + x^4 + x^5 + x^6) = x\frac{1 - x^6}{1-x}$$

and the total of the four dice rolls by the generating function

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^4 = x^4\left(\frac{1 - x^6}{1-x}\right)^4$$

We could multiply out the function and find the answer (the coefficient of x^{12}), but we can be a bit more clever by manipulating the function first.

First we can cancel out the x^4 factor, so we are looking for the coefficient of x^8 in

$$\left(\frac{1-x^6}{1-x}\right)^4$$

We may apply the binomial theorem to obtain

$$\begin{aligned}(1-x^6)^4 &= \binom{4}{0}(-x^6)^0 + \binom{4}{1}(-x^6)^1 + \binom{4}{2}(-x^6)^2 + \binom{4}{3}(-x^6)^3 + \binom{4}{4}(-x^6)^4 \\ &= 1 - 4x^6 + 12x^{12} - 4x^{18} + x^{24} \qquad (1)\end{aligned}$$

Now also note that

$$(1-x)^{-4} = \frac{1}{(1-x)^4} = (1 + x + x^2 + \dots)^4 \qquad (2)$$

and similarly to our first example today, the coefficient of the x^n term is $\binom{n+3}{3}$.

So, if we are looking for the coefficient of x^8 in $\left(\frac{1-x^6}{1-x}\right)^4$, the terms with powers x^{12} and higher in (1) will not contribute to this. The only way of getting x^8 is to either take the 1 term from (1) and the $\binom{11}{3}x^8$ term from () or to take the $-4x^6$ term from (1) and the $\binom{5}{3}x^2$ term from (3) for a total coefficient of

$$\binom{11}{3} - 4\binom{5}{3}$$

4 The generalized binomial theorem (Chapter 8.3 in [KT17])

Recall the following corollary of the Binomial Theorem:

Corollary 4.1. *If $p \geq 1$ is a positive integer, then*

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n.$$

Note that this means that the generating function for the number of n-subsets of a p-element set is $(1+x)^p$.

The right-hand side $(1+x)^p$ makes sense for p any real number, but binomial coefficients are only defined for $p, n \in \mathbb{Z}_{\geq 0}$.

Definition 4.2. *The **generalized binomial coefficient**, for $n \geq 0$, $p \in \mathbb{R}$ is*

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!},$$

note that $\binom{p}{0} = 1$.

Note that this definition agrees with our definition of binomial coefficients for $p \in \mathbb{Z}_{\geq 0}$.

Example 4.3.

$$\binom{-2}{5} = \frac{(-2)(-3)(-4)(-5)(-6)}{5!} = -6$$

It turns out that with this definition, the Binomial Theorem generalizes

Theorem 4.4 (Theorem 8.9. in [KT17])**.** *For all $p \in \mathbb{R}, p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$

As a sanity check, note that this does not contradict the Corollary above, since if $p \geq 1$ is a positive integer and $n > p$, then $\binom{p}{n} = 0$.

Why is Theorem 4.4 useful? If we have a generating function of the form $f(x) = (1 + ax)^p$, then the coefficient of x^n is $\binom{p}{n} a^p$ (for $p \in \mathbb{R}$). The case $p = -1$ will be particularly useful for us when working with various generating functions.

Corollary 4.5. *If*

$$f(x) = \frac{c}{(1 + ax)^m}$$

for some constants a, c, then the coefficient of x^n in the series expansion of $f(x)$ is

$$c\binom{-m}{n} a^n = ca^n \frac{(-m)(-m-1)\dots(-m-n+1)(-m-n+1)}{n!} = ca^n(-1)^n \frac{(n+m-1)(n+m-2)\cdots(m+1)(m)}{n!}.$$

In particular, if $m = -1$, then the coefficient of x^n in the series expansion of $f(x)$ is

$$c\binom{-1}{n} a^n = ca^n \frac{(-1)(-2)\dots(-n+1)(-n)}{n!} = ca^n(-1)^n = c(-a)^n.$$

Using the method of partial fractions, we can apply Theorem 4.4 to more functions.

Exercise 4.6 (Example 8.1. in [Mor17])**.** *Suppose we have a generating function*

$$f(x) = \frac{1 + x}{(1 - 2x)(2 + x)}$$

What is the coefficient of x^n?

Solution: We want to write

$$f(x) = \frac{1 + x}{(1 - 2x)(2 + x)} = \frac{A}{1 - 2x} + \frac{B}{2 + x}.$$

for some $A, B \in \mathbb{R}$. Note that this leads to

$$\frac{A(2 + x) + B(1 - 2x)}{(1 - 2x)(2 + x)} = \frac{1 + x}{(1 - 2x)(2 + x)}.$$

and therefore

$$A(2 + x) + B(1 - 2x) = 1 + x.$$

Since this is an equality between two polynomials, it should hold for specific values of x. If we let $x = -2$, we see that

$$5B = -1,$$

or, $B = \frac{-1}{5}$, and if we let $x = \frac{1}{2}$, we see that

$$\frac{5}{2}A = \frac{3}{2}$$

or, $A = \frac{3}{5}$. Therefore we have

$$f(x) = \frac{\frac{3}{5}}{1 - 2x} - \frac{\frac{1}{5}}{2 + x}.$$

We can already use Corollary 4.5 on the first term, but the $(2 + x)$ denominator is still a problem. So we rewrite

$$\frac{\frac{1}{5}}{2 + x} = \frac{\frac{1}{10}}{1 + \frac{x}{2}}.$$

Then we can apply Corollary 4.5 to both terms, and find that the coefficient of x^n in $f(x)$ is

$$\frac{3}{5}2^n - \frac{1}{10}\left(\frac{-1}{2}\right)^n$$

Example 4.7 (Example 8.6 in [KT17]). *Find the number of integer solutions to the equation*

$$x_1 + x_2 + x_3 = n$$

with $x_1 \geq 0$ *even,* $x_2 \geq 0$ *and* $0 \leq x_3 \leq 2$.

This problem would be very difficult without using generating functions, as we don't have a good way to represent this computation as stars and bars.

Solution: The generating function for the problem is easy to write down, we just multiply the generating functions for the terms together to get

$$f(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \left(1 + x + x^2\right) = \frac{1+x+x^2}{(1-x^2)(1-x)}.$$

We again want to rewrite this as sums of simpler rational functions. Since the denominator has a double root, we want to solve

$$\frac{1+x+x^2}{(1-x^2)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

for A, B, C constants. We again clear the denominators to get

$$1 + x + x^2 = A(1-x)^2 + B(1-x^2) + C(1+x)$$

substituting $x = 1$ gives

$$3 = 2C,$$

or, $C = \frac{3}{2}$. substituting $x = -1$ gives

$$1 = 4A,$$

or, $A = \frac{1}{4}$. Then we can compare the x^2 terms on both sides to obtain

$$1 = A - B$$

and we get $B = -\frac{3}{4}$.

So we may write

$$f(x) = \frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}$$

so the coefficient of x^n in $f(x)$ is

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2}.$$

(note that this is an integer).

5 Generating functions and recursion

Example 5.1 (Example 8.6. in [Mor17]). *Consider the recursively defined sequence* $a_0 = 2$, *and for every* $n \geq 1, a_n = 3a_{n-1} - 1$. *Find an explicit formula for* a_n *in terms of* n.

Solution: The sequence a_n has generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$. How can we use the fact that $a_m = 3a_{m-1} - 1$? Note that

$$\begin{array}{rllllll} a(x) = & a_0 & + a_1 x & + a_2 x^2 & + \dots & + a_m x^m & + \dots \\ -3xa(x) = & & - 3a_0 x & - 3a_1 x^2 & - \dots & - 3a_{m-1} x^m & - \dots \\ a(x) - 3xa(x) = & a_0 & - x & - x^2 & - \dots & - x^m & - \dots \end{array}$$

Therefore, we have the following equality

$$(1-3x)a(x) = 3 - (1 + x + x^2 + \dots)$$

and since $(1 + x + x^2 + \dots) = \frac{1}{1-x}$, we have the equality between generating functions

$$(1-3x)a(x) = 3 - \frac{1}{1-x},$$

or, equivalently,

$$a(x) = \frac{3}{1-3x} - \frac{1}{(1-x)(1-3x)}$$

we can use the method of partial fractions to rewrite this as

$$a(x) = \frac{\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}.$$

The coefficient of x^n is therefore

$$a_n = \frac{1}{2} + \frac{3}{2}3^n.$$

We can check the first few terms of this sequence, in particular,

$$\begin{aligned} a_0 &= \frac{1}{2} + \frac{3}{2} = 2 \\ a_1 &= \frac{1}{2} + \frac{3}{2}3 = 5 \\ a_2 &= \frac{1}{2} + \frac{3}{2}3^2 = 14 \end{aligned}$$

Which agrees with the recurrence relation $a_n = 3a_{n-1} - 1$.

Example 5.2 (Example 1.2 in [Wil90])**.** *A certain sequence of numbers $a_0, a_1, \dots$ satisfies the conditions*

$$a_{n+1} = 2a_n + n \qquad (n \geq 0; a_0 = 1).$$

Find the sequence.

Solution: The sequence a_n has generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$. How can we get an identity similar to the one in Example 5.1? We could write something similar, but we'll use a slightly different looking, but equivalent method this time. If we multiply both sides by x^n, we get

$$a_{n+1}x^n = 2a_n x^n + nx^n,$$

then we can sum over all n **where the identity is valid** (which is $n \geq 0$ in this case). So we get

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} \left(2a_n x^n + nx^n\right). \tag{3}$$

The next step is to try to bring in $a(x)$ into equation (3). The left hand side of (3) is

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+1}x^n &= a_1 + a_2 x + a_3 x^2 + \dots \\ &= \frac{a_1 x + a_2 x^2 + a_3 x^3 + \dots}{x} \\ &= \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots}{x} - \frac{a_0}{x} \\ &= \frac{a(x)}{x} - \frac{1}{x} && \text{since } a_0 = 1. \\ &= \frac{a(x) - 1}{x}. \end{aligned}$$

Moving on to the right hand side of (3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(2a_nx^n + nx^n) &= 2\sum_{n=0}^{\infty}a_nx^n + \sum_{n=0}^{\infty}nx^n \\ &= 2a(x) + \sum_{n=0}^{\infty}nx^n.\end{aligned}$$

So we need to figure out how to identify the function that has the series $\sum_{n=0}^{\infty} nx^n$. Notice that this looks almost like

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

except that the coefficient of x^n is n, not $n-1$, so the coefficients are shifted by one. How can we fix this? We can factor out a single power of x from the entire series to get

$$\sum_{n=0}^{\infty} nx^n = x\sum_{n=0}^{\infty} nx^{n-1}.$$

It is important to note that this is only valid because our $n=0$ term is 0 (otherwise we'd need adjust the indexing to start at $n=1$). Putting these together, we see that

$$\sum_{n=0}^{\infty} nx^n = x\sum_{n=0}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

Going back to equation (3), we bring in what we found for the left and right hand sides to get

$$\begin{aligned}\sum_{n=0}^{\infty}a_{n+1}x^n &= \sum_{n=0}^{\infty}(2a_nx^n + nx^n) \\ \frac{a(x)-1}{x} &= 2a(x) + \frac{x}{(1-x)^2}.\end{aligned}$$

We can now solve for $a(x)$ to get

$$a(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}.$$

It still remains to actually determine the sequence a_n, which we now can interpret as the coefficients of the Taylor expansion of $a(x)$. Since $a(x)$ is a rational function, we can use partial fractions

$$a(x) = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x}.$$

We find that $A=-1, B=0, C=2$, so we have

$$a(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}.$$

We can then use Corollary 4.5 and what we found earlier for the series $\frac{1}{(1-x)^2}$ to get

$$a_n = 2^{n+1} - n - 1.$$

We can check that it satisfies the recuurence, note that

$$a_{n+1} = 2^{n+2} - n - 2.$$

and

$$2a_n + n = 2\left(2^{n+1} - n - 1\right) + n = 2^{n+2} - n - 2.$$

References

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[Wil90] Herbert S. Wilf. *Generatingfunctionology*. Academic Press, 1990. Available at https://www.math.upenn.edu/~wilf/DownldGF.html. 6