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Logarithmic functional and reciprocity laws

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Abstract. In this paper, we give a short survey of results related to the reciprocity laws over the field $\mathbb{C}$. We announce a visual topological proof of Parshin's multidimensional reciprocity laws over $\mathbb{C}$. We introduce the logarithmic functional, whose argument is an n-dimensional cycle in the group $(\mathbb{C}^*)^{n+1}$. It generalizes the usual logarithm, which can be considered as the zero-dimensional logarithmic functional. It also generalizes the one-dimensional logarithmic functional that is a natural extension of the functional introduced by Beilinson for a topological proof of the Weil reciprocity law over $\mathbb{C}$.

1. One-dimensional case

1.1. Weil reciprocity law. Let Γ be a complete connected complex one-dimensional manifold (in other words, Γ is an irreducible complex algebraic curve). A *local parameter* u near a point $a \in \Gamma$ is an arbitrary meromorphic function, whose order at a is equal to one. The local parameter u is a coordinate function in a small neighborhood of a. Let φ be a meromorphic function on Γ and let $\sum_{k \leq m} c_m u^m$ be its Laurent expansion at a. The *leading monomial* χ of φ is the first nonzero term in the expansion, i.e. $\chi(u) = c_k u^k$. The leading monomial is defined for any meromorphic function φ not identically equal to zero. For each pair of meromorphic functions f, g on a curve Γ not identically equal to zero and each point $a \in \Gamma$, one defines *the Weil symbol* $[f, g]_a$. It is a nonzero complex number given by the formula

$$[f, g]_a = (-1)^{nm} a_m^n b_n^{-m},$$

where $a_m u^m$ and $b_n u^n$ are the leading monomials of the functions f and g at a, with respect to the parameter u. The Weil symbol is defined with the help of the parameter u but *it does not depend on the choice of* u. By definition, the *Weil symbol depends multiplicatively on functions* f *and* g. The multiplicativity with respect to f means that if $f = f_1 f_2$, then $[f, g]_a = [f_1, g]_a [f_2, g]_a$. The multiplicativity with respect to g is defined similarly. The Weil symbol of functions f, g can differ from 1 only at points in the supports of the divisors of f and g.

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The Weil reciprocity law. *The product of the Weil symbols $[f,g]_a$ over all points a of the curve Γ is equal to one*

$$\prod_{a\in X}[f,g]_a=1.$$

Example. Take the Riemann sphere for Γ, an affine coordinate function x for f, and a polynomial $P=a_nx^n+\cdots+a_kx^k$ of degree n for g. By the reciprocity law,

$$\prod x(a)=[x,P]_0^{-1}[x,P]_\infty^{-1}=(-1)^{-k}a_k^{-1}(-1)^na_n=(-1)^{n-k}a_n/a_k,$$

where the product is over all nonzero roots a of P. This formula coincides with the Vieta formula.

1.2. Toric Surfaces and the reciprocity law (see [1]). Consider a compact (possibly singular) toric surface M. Let D be a zero-dimensional positive divisor in the union of one-dimensional orbits. Is there an algebraic curve on the surface M that does not pass through zero-dimensional orbits and intersects one-dimensional orbits at the given divisor D?

Let us fix an orientation in the plane of one-parameter subgroups of $(\mathbb{C}^*)^2$. Thus we fix a parameterization $\pi_j:\mathbb{C}^*\to M_j$ of each one-dimensional orbit M_j in M. Consider the map $\pi:\bigcup_j M_j\to\mathbb{C}^*$, whose restriction to M_j equals to π_j^{-1}.

Theorem. *If $D=\sum k_ia_i$ is the divisor of the intersection of a curve not passing through zero-dimensional orbits of M with the union of one-dimensional orbits of M, then*

$$\prod(-\pi(a))^{k_i}=1$$

Sketch of the proof. Each curve in $(\mathbb{C}^*)^2$ is given by an equation $P=0$, where P is a Laurent polynomial. Let Δ be the Newton polygon of P. On each side $\mathbf{n}_j$ of Δ, a polynomial $P_{\mathbf{n}_j}$ in one variable is written. By the Vieta formula, the product of all nonzero roots of all polynomials $P_{\mathbf{n}_j}$ is equal to 1.

The reciprocity law follows from the previous theorem (see [1]).

1.3. Topological proof of the reciprocity law (see [2 - 4]). Consider a complex algebraic curve Γ and a pair f,g of nonzero meromorphic functions on Γ. Let A be the union of the supports of the principle divisors (f), (g), and let U be $\Gamma\setminus A$. With the pair f,g, Beilinson associated a certain cohomology class $[f,g]\in H^1(U,\mathbb{C}^*)$. One can define this class using the Beilinson Integral.

Definition. *The Beilinson Integral against a loop $\gamma:I\to U$, $\gamma(0)=\gamma(1)$, for a pair of analytic functions $f:U\to\mathbb{C}^*$, $g:U\to\mathbb{C}^*$* is an element $I_\gamma(f,g)$ of the group $\mathbb{C}/\mathbb{Z}$ defined by the formula

$$I_\gamma(f,g)=\left(\frac{1}{2\pi i}\right)^2\int_I\ln(\gamma^*f)\frac{d(\gamma^*g)}{(\gamma^*g)}-\frac{1}{2\pi i}\deg_\gamma(f)\ln g(\gamma(1)),$$

where $\ln(\gamma^*f)$ is a continuous branch of the multi-valued function $\ln(\gamma^*f)$ over the interval $0<t<1$, and $\deg_\gamma(f)$ is the mapping degree of the map $\dfrac{f}{|f|}:\gamma(I)\to S^1$.

One can prove that the Beilinson integral has the following properties. It is invariant under orientation preserving reparameterizations of the loop γ. It is skew symmetric, i.e. $I_\gamma(f,g) = -I_\gamma(g,f)$, and additive, i.e. $I_\gamma(f_1f_2,g) = I_\gamma(f_1,g) + I_\gamma(f_2,g)$; $I_\gamma(f,g_1g_2) = I_\gamma(f,g_1) + I_\gamma(f,g_1)$. On the diagonal, it is related to the mapping degree: $I_\gamma(f,f) = \frac{1}{2}\deg_\gamma(f)$.

One can slightly extend the previous definition and define the Beilinson integral $I_\gamma(f,g)$ against a linear combination $\gamma = \sum k_i\gamma_i$ of loops γ_i with integer coefficients k_i.

THEOREM. *The Beilinson integral $I_\gamma(f,g)$ depends only on the homology class of the cycle $\gamma = \sum k_i\gamma_i$ in U and defines an element in $H^1(U,\mathbb{C}/\mathbb{Z})$.*

THEOREM. *The number $[f,g,\gamma] = \exp(2\pi i I_\gamma(f,g))$ depends only on the homology class of the cycle $\gamma = \sum k_i\gamma_i$ in U and defines an element in $H^1(U,\mathbb{C}^*)$. It is skew-symmetric $[f,g,\gamma] = [g,f,\gamma]^{-1}$ and multiplicative $[f_1f_2,g,\gamma] = [f_1,g,\gamma]\cdot[f_2,g,\gamma]$; $[f,g_1g_2,\gamma] = [f,g_1,\gamma][f,g_2,\gamma]$.*

THEOREM. *Consider a small ball B_a centered at a point $a \in X$. Let γ be its boundary $\gamma = \partial B$. Then $[f,g,\gamma]$ is equal to the Weil symbol $[f,g]_a$.*

Now let us give a topological proof of the reciprocity law. Let A be the union of supports of principle divisors (f), (g), and let U be $\Gamma \setminus A$. Let B be the union of small balls B_a centered at all points $a \in A$ and let $\gamma = \sum_{a\in A}\partial B_a$ be the boundary of the domain B. Then $\gamma = 0$ in $H_1(U,\mathbb{Z})$ and $[f,g,\gamma] = \prod_{a\in A}[f,g]_a = 1$. The reciprocity law is proved.

2. Product of roots of a system of equations with generic Newton polyhedra (see [5])

Consider a system of equations

$$P_1 = \cdots = P_n = 0 \tag{1}$$

in $(\mathbb{C}^*)^n$, where $P_1,\dots,P_n$ are Laurent polynomials. Let $\Delta_1,\dots,\Delta_n$ be the Newton polyhedra of $P_1,\dots,P_n$.

Problem: *Compute the product in the group $(\mathbb{C}^*)^n$ of roots of system (1) assuming that the collection of Newton polyhedra $\Delta_1,\dots,\Delta_n$ is generic.*

2.1. Developed sets of polyhedra. Let $\Delta_1,\dots,\Delta_n$ be convex polyhedra in $\mathbb{R}^n$, and let Δ be their Minkowski sum. Each face Γ of the polyhedron Δ can be uniquely represented as a sum

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n,$$

where Γ_i is a face of Δ_i.

A collection of n polyhedra $\Delta_1,\dots,\Delta_n$ is called *developed* if for each face Γ of the polyhedron Δ, at least one of the terms Γ_i in its decomposition is a vertex.

For a developed collection of polyhedra $\Delta_1,\dots,\Delta_n$, a map $f : \partial\Delta \to \partial\mathbb{R}^n_+$ of the boundary of $\Delta = \sum\Delta_i$ into the boundary of the positive octant is called *characteristic* if the component f_i of the map $f = (f_1,\dots,f_n)$ vanishes precisely on the faces Γ, for which the i-th term Γ_i in the decomposition is a point (a vertex of the polyhedron Δ_i). The preimage of the origin under the characteristic map is precisely the set of all vertices of the polyhedron Δ.

The *combinatorial coefficient* C_A of a vertex A of Δ is the local degree of the germ

$$f\colon (\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$$

of the characteristic map restricted to the boundary $\partial\Delta$ of Δ.

2.2. Parshin symbols see [5 - 9]. Consider $n+1$ monomials $c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}$ in n complex variables, where $c_i \in \mathbb{C}^*$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{k}_i \in (\mathbb{Z})^n, \mathbf{k}_i = (k_{i,1}, \dots, k_{i,n})$, $c_i\mathbf{x}^{\mathbf{k}_i} = c_i x_1^{k_{i,1}} \cdot \ldots \cdot x_n^{k_{i,n}}$. The *Parshin Symbol* $[c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}]$ of the sequence $c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}$ is equal by definition to

$$(-1)^{D(\mathbf{k}_1,\dots,\mathbf{k}_{n+1})} c_1^{-\det(\mathbf{k}_2,\dots,\mathbf{k}_{n+1})} \dots c_{n+1}^{(-1)^{n+1}\det(\mathbf{k}_1,\dots,\mathbf{k}_n)} =$$

$$= (-1)^{D(\mathbf{k}_1,\dots,\mathbf{k}_{n+1})} \exp\left(-\det \begin{pmatrix} \ln c_1 & k_{1,1} & \dots & k_{1,n+1} \\ \vdots & \vdots & & \vdots \\ \ln c_{n+1} & k_{n+1,1} & \dots & k_{n+1,n+1} \end{pmatrix} \right),$$

where $D : (\mathbb{Z}^n)^{n+1} \to \mathbb{Z}/2\mathbb{Z}$ is a function (see [9]) with the following properties. The function D depends only on the images $\pi(\mathbf{k}_1), \dots, \pi(\mathbf{k}_{n+1}) \in (\mathbb{Z}/2\mathbb{Z})^n$ of the exponents $\mathbf{k}_1, \dots, \mathbf{k}_{k+1}$ under the natural projection $\pi : (\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$. It is the only nonzero multilinear function of $n+1$ vectors in the n-dimensional space over the field $\mathbb{Z}/2\mathbb{Z}$ that is invariant under all linear transformations and vanishes whenever the rank of $n+1$ vectors is less than n.

Example. The Parshin symbol $[c_1x^{k_1}, c_2x^{k_2}]$ of two monomials $c_1x^{k_1}$, $c_2x^{k_2}$ in one variable x equals to $(-1)^{k_1k_2}c_1^{-k_2}c_1^{k_1}$. Thus it is equal to the Weil symbol $[c_1x^{k_1}, c_2x^{k_2}]_0$ of these monomials at the origin $x = 0$.

By definition, the Parshin symbol is skew-symmetric, for example,

$$[c_1\mathbf{x}^{\mathbf{k}_1}, c_2\mathbf{x}^{\mathbf{k}_2}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}] = [c_2\mathbf{x}^{\mathbf{k}_2}, c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}]^{-1},$$

and multiplicative, for example, if $c_1\mathbf{x}^{\mathbf{k}_1} = a_1b_1\mathbf{x}^{\mathbf{l}_1+\mathbf{m}_1}$, then

$$[c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}] = [a_1\mathbf{x}^{\mathbf{l}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}][b_1\mathbf{x}^{\mathbf{m}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}].$$

2.3. The value of a character at the product of roots. Let $\chi_{\mathbf{k}} : (\mathbb{C}^*)^n \to \mathbb{C}^*$ be the character corresponding to a point $\mathbf{k} \in \mathbb{Z}^n$, i.e. for $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ the character $\chi_{\mathbf{k}}(\mathbf{x})$ is equal to $x_1^{k_1} \dots x_n^{k_n}$.

With every vertex A of the polyhedron $\Delta = \Delta_1 + \dots + \Delta_n$, where Δ_i is the Newton polyhedron of a Laurent polynomial P_i, we associate a number $[P_1, \dots, P_n, \chi_{\mathbf{k}}]_A \in \mathbb{C}^*$ in the following way: let $A = A_1 + \dots + A_n$ be the decomposition of the vertex $A \in \Delta$, $A_i \in \Delta_i$. Assume that the vertex A_i of the polyhedron Δ_i corresponds to a monomial $c_i\mathbf{x}^{\mathbf{k}_i}$ of the polynomial P_i. Then the number $[P_1, \dots, P_n, \chi_{\mathbf{k}}]_A$ is by definition the Parshin symbol $[c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_n\mathbf{x}^{\mathbf{k}_n}, \chi_{\mathbf{k}}]$.

Theorem. *For a system of equations (1), the value of the character $\chi_{\mathbf{k}}$ at the product $M(P_1, \dots, P_n)$ of roots is given by*

$$\chi_{\mathbf{k}}(M(P_1, \dots, P_n)) = \prod_{A \in \Delta} ([P_1, \dots, P_n, \chi_{\mathbf{k}}]_A)^{(-1)^n C_A},$$

where the product is over all vertices A of the polyhedron $\Delta = \Delta_1 + \dots + \Delta_n$, and C_A is the combinatorial coefficient at A.

As an application of the theorem, one can compute all coordinates in $(\mathbb{C}^*)^n$ of the product $M(P_1,\dots,P_n)$ of roots: each coordinate x_i can be considered as the character $\chi_{\mathbf{k}}$ for $\mathbf{k}=e_i$, where the vector e_i is the i-th vector in the standard basis of the lattice $(\mathbb{Z})^n$. The proof of the theorem (see [5]) is based on simple geometry and does not use Parshin theory.

3. Multidimensional case

Parshin generalized the Weil reciprocity law to a multidimensional case (see [6 - 9]). Here we discuss this result and its topological proof due to Brylinski–McLaughlin (see [10]).

3.1. Generalized Points, Parameters, Symbols, Flags and Reciprocity Laws (see [6 - 8]). Let X be a complete irreducible complex n-dimensional algebraic variety (possibly very singular). A sequence $Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-1}} Y_n \xrightarrow{\pi_n} X$ consisting of complete normal irreducible i-dimensional algebraic varieties Y_i with $i=0,\dots,n$, equipped with a collection of maps $\pi_0,\dots\pi_n$ is called *a generalized point of the variety* X if for $i=0,\dots,n-1$, the map $\pi_i: Y_i \to Y_{i+1}$ is a normalization of the image $\pi_i(Y_i)\subset Y_{i+1}$ and $\pi_n: Y_n\to X$ is a normalization of X. We identify two generalized points $G_1=(Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-1}} Y_n \xrightarrow{\pi_n} X)$ and $G_2=(Z_0 \xrightarrow{\rho_0} Z_1 \xrightarrow{\rho_1} \dots \xrightarrow{\rho_{n-1}} Z_n \xrightarrow{\rho_n} X)$ if for $i=0,\dots,n$, there are isomorphisms $\tau_i: Z_i\to Y_i$ such that $\pi_i\circ\tau_i=\tau_{i+1}\circ\rho_i$.

Let us give an inductive definition for a set of parameters near a generalized point. A collection of rational functions $u_1,\dots,u_n$ on X is called *a set of parameters near a generalized point* $G=(Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-1}} Y_n \xrightarrow{\pi_n} X)$ if the following conditions hold:

1) each of the rational functions $\pi_n^*u_1,\dots,\pi_n^*u_n$ on Y_n has no poles on $X_{n-1}=\pi_{n-1}(Y_{n-1})$;

2) the principal divisor $(\pi_n^*u_n)$ in the normal variety Y_n contains the subvariety X_{n-1} with coefficient 1;

3) if $n>1$, then the set of restrictions of $\pi_n^*u_1,\dots,\pi_n^*u_{n-1}$ to X_{n-1} is a set of parameters for the generalized point $\tilde{G}=(Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-2}} Y_{n-1} \xrightarrow{\pi_{n-1}} X_{n-1})$ on the $(n-1)$-dimensional variety X_{n-1} .

With a rational function f and a generalized point G one can associate the *leading monomial* f_G using a set of parameters near G. Let us give an inductive definition of the leading monomial. Let the order of the function π_n^*f at X_{n-1} be k_n. Then the restriction φ of $(\pi_n^*f)u^{-k_n}$ to X_{n-1} is well-defined. Consider the generalized point $\tilde{G}=(Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-2}} Y_{n-1} \xrightarrow{\pi_{n-1}} X_{n-1})$ of the $(n-1)$-dimensional variety X_{n-1}. Let $c(\pi_n^*u_1)^{k_1}\dots(\pi_n^*u_{n-1})^{k_{n-1}}$ be the leading monomial of φ at the generalized point $\tilde{G}$ with parameters $(\pi_n^*u_1),\dots,(\pi_n^*u_{n-1})$. Then, by definition, the leading monomial f_G is equal to $cu_1^{k_1}\dots u_{n-1}^{k_{n-1}}u_n^{k_n}$.

Let $f_1,\dots,f_{n+1}$ be a sequence of $(n+1)$ rational functions on an n-dimensional algebraic variety X. With a generalized point G, one can associate the *Parshin symbol* $[f_1,\dots,f_{n+1}]_G$: by definition, it is equal to the Parshin symbol of a sequence of leading monomials of the functions with respect to a set of parameters near the generalized point G. One can prove that the *Parshin symbol is independent of a set of parameters* and depends only on the sequence of rational functions $f_1,\dots,f_{n+1}$ and on the generalized point G.

Consider a flag $F = (C_0 \subset C_1 \subset \cdots \subset C_{n-1})$ in X consisting of complete irreducible k-dimensional algebraic subvarieties C_k of X with $k = 0, 1, \ldots, n-1$. A generalized point $G = (Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{n-1}} Y_n \xrightarrow{\pi_n} X)$ is *a generalized point over the flag* F if for $j = 0, \ldots, n-1$, the image $\tilde{\pi}_j(Y_j)$ of Y_j under the map $\tilde{\pi}_j = \pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_j$ is equal to C_j. Over each flag F, there are finitely many different generalized points. Consider a flag $F = C_0 \subset \cdots \subset C_{n-1}$ on X and take a collection $f_1, \ldots, f_{n+1}$ of rational functions on X. The *Parshin symbol* $[f_1, \ldots, f_{n+1}]_F$ *of the collection* $f_1, \ldots, f_{n+1}$ *at the flag* F is by definition the product of $\prod [f_1, \ldots, f_{n+1}]_G$ over all generalized points G over the flag F.

Example. Let X be an irreducible algebraic curve. With a generalized point $G = (Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} X)$ in X, one can associate the flag $F = (a)$ in X, where $a = \pi_1 \circ \pi_0(Y_0)$, and the point $b = \pi_0(Y_0)$ on the normalization Y_1 of the curve X. A rational function u on X is a parameter near G if $\pi_1^* u$ is a parameter near b on Y_1. For a pair of rational functions f_1, f_2 on X, the Parshin symbol $[f_1, f_2]_G$ coincides with the Weil symbol $[\pi_1^* f_1, \pi_1^* f_2]_b$. The Parshin symbol $[f_1, f_2]_F$ at the flag F is equal to the product $\prod [\pi_1^* f_1, \pi_1^* f_2]_c$ over all points $c \in \pi_1^{-1} a$.

Fix a flag $L = (C_0 \subset \cdots \subset C_{n-1})$ on X. For each $0 < i < n$, denote by $\Psi^i(L)$ the set of all flags $F = (\tilde{C}_0 \subset \cdots \subset \tilde{C}_{n-1})$, where $\tilde{C}_j = C_j$ if $j \neq i$, and $\tilde{C}_i$ is any i-dimensional irreducible subvariety such that $C_{i-1} \subset \tilde{C}_i \subset C_{i+1}$. Denote by $\Psi^0(L)$ the set of all flags $F = \tilde{C}_0 \subset C_1 \cdots \subset C_{n-1}$ such that $\tilde{C}_0$ is a point in C_1.

Parshin's reciprocity laws (see [6 - 8]). *Fix a collection* $f_1, \ldots, f_{n+1}$ *of rational functions on* X*, a flag* L *in* X *and a number* $0 \le i < n$*. Then the symbol* $[f_1, \ldots, f_{n+1}]_F$ *is different from 1 for only finitely many flags* $F \in \Psi^i(L)$*, and the following relation holds*

$$\prod_{F \in \Psi^i(L)} [f_1, \ldots, f_{n+1}]_F = 1.$$

3.2. The Brylinski–McLaughlin topological version of the Parshin theory (see [10]). Consider a sequence $f_1, \ldots, f_{n+1}$ of $(n+1)$ rational functions on an n-dimensional complete complex algebraic variety X. Let A be the union of the supports of the principle divisors $(f_1), \ldots, (f_{n+1})$ and of the singular locus $S(X)$ of the variety X. Denote by U the domain $X \setminus A$.

Brylinski and McLaughlin defined a certain cohomology class $(f_1, \ldots, f_{n+1}) \in H^n(U, \mathbb{C}^*)$. The class $(f_1, \ldots, f_{n+1})$ is skew symmetric in $f_1, \ldots, f_{n+1}$ and is multiplicative in each argument.

Let $F = (C_0 \subset C_1 \subset \cdots \subset C_{n-1})$ be a flag of irreducible complete algebraic subvarieties of X such that $\dim C_k = k$. With the flag F, Brylinski and McLaughlin associated a *flag-localized homology class* $\gamma_F \in H_n(U, \mathbb{Z})$. They had proved the following results.

Theorem (Topological reciprocity laws). *Fix a flag* L *in* X *and a number* $0 \le i < n$*. Then the flag-localized homology class* $\gamma_F \in H_n(U, \mathbb{Z})$ *is different from zero for only finitely many flags* $F \in \Psi^i(L)$*, and in the group* $H_n(U, \mathbb{Z})$*, the following relation holds*

$$\sum_{F \in \Psi^i(L)} \gamma_F = 0.$$

THEOREM. *The Parshin symbol* $[f_1, \dots, f_{n+1}]_F$ *can be obtained by the pairing of the cohomology class* $(f_1, \dots, f_{n+1}) \in H^n(U, \mathbb{C}^*)$ *with the flag-localized homology class* $\gamma_F \in H_n(U, \mathbb{Z})$.

Using these results one can immediately obtain Parshin's reciprocity laws. So Brylinski and McLaughlin gave a topological proof of the multidimensional reciprocity laws over complex numbers and found a topological generalization of Parshin symbols. Their topological constructions make a heavy use of sheaf theory and are not visual at all.

The search for a formula for the product of the roots of a system of equations (see [5]) convinced me that, over complex number, there should be an intuitive geometric explanation of the Parshin symbols and reciprocity laws. The multidimensional logarithmic functional provides such an explanation.

4. A Logarithmic Functional (see [11])

4.1. Definitions and examples. Consider the group $(\mathbb{C}^*)^{n+1}$ with coordinate functions $x_1, \dots, x_{n+1}$. The group $(\mathbb{C}^*)^{n+1}$ is homotopy equivalent to the torus $T^{n+1} \subset (\mathbb{C}^*)^{n+1}$ defined by equations $|x_1| = \dots = |x_{n+1}| = 1$. On the group $(\mathbb{C}^*)^{n+1}$, there is a remarkable $(n+1)$-form

$$\omega = \left(\frac{1}{2\pi i}\right)^{n+1} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_{n+1}}{x_{n+1}}.$$

Restriction of the form ω to the torus T^{n+1} is the real $(n+1)$-form

$$\omega|_{T^{n+1}} = \frac{1}{(2\pi)^n} d(\arg x_1) \wedge \dots \wedge d(\arg x_{n+1}).$$

The integral of ω over the torus T^{n+1} oriented by the form $\omega|_{T^{n+1}}$ is equal to 1.

Denote by E the subset of $(\mathbb{C}^*)^{n+1}$ consisting of all points $(x_1, \dots, x_{n+1})$, one of whose coordinates is equal to 1, i.e. $(x_1, \dots, x_{n+1}) \in E \Leftrightarrow \{\exists j, 1 \le j \le n+1\} : \{x_j = 1\}$.

Let X be an n-dimensional simplicial complex and let $\gamma = \sum k_j \Delta_j$, where $k_j \in \mathbb{Z}$ and Δ_j is an oriented n-dimensional cell in X, be an n-dimensional cycle, i.e. $\partial\gamma = 0$. Consider a piecewise-smooth mapping $\mathbf{f} : X \to (\mathbb{C}^*)^{n+1}$, $\mathbf{f} = (f_1, \dots, f_{n+1})$, of the complex X into the group $(\mathbb{C}^*)^{n+1}$.

The *logarithmic functional* is the functor that takes a map $\mathbf{f} : X \to (\mathbb{C}^*)^{n+1}$ and an n-dimensional cycle $\gamma = \sum k_j \Delta_j$ on X to the element $\ln(\mathbf{f}, \gamma)$ of $\mathbb{C}/\mathbb{Z}$ defined by the formula

$$\ln(\mathbf{f}, \gamma) = \int_\sigma \omega,$$

where σ is an $(n+1)$-dimensional chain in $(\mathbb{C}^*)^{n+1}$, whose boundary $\partial\sigma$ is equal to the difference of the image $\mathbf{f}_*(\gamma)$ of γ under the map $\mathbf{f}_*$ and some cycle γ_1 lying in E.

The logarithmic functional has the following obvious properties:

1) The value of the logarithmic functional is a well-defined element of the group $\mathbb{C}/\mathbb{Z}$.

2) The logarithmic functional depends skew-symmetrically on the components of the map $\mathbf{f}$, for example

$$\ln(f_1, f_2, \dots, f_{n+1}, \gamma) = -\ln(f_2, f_1, \dots, f_{n+1}, \gamma).$$

3) The logarithmic functional depends multiplicatively on the components of the map $\mathbf{f}$, for example,

$$\ln(\varphi_1\psi_1, f_2, \dots, f_{n+1}, \gamma) = \ln(\varphi_1, f_2, \dots, f_{n+1}, \gamma) + \ln(\psi_1, f_2, \dots, f_{n+1}, \gamma)$$

Example (logarithmic function as logarithmic functional). Let X be a point $\{a\}$ and γ the point $\{a\}$ with coefficient 1. Then for a map $\mathbf{f} : X \to \mathbb{C}^*$, we have

$$\ln(\mathbf{f}, \gamma) = \frac{1}{2\pi i} \ln f(a),$$

where $f = \mathbf{f}$.

Example (The Beilinson integral as the logarithmic functional [3]). Consider a complex algebraic curve X and two nonzero meromorphic functions f and g on X. Let A be the union of the divisors (f) and (g), and let U be $X \setminus A$. Take a loop γ in U. Then for a map $(f, g) : \gamma \to (\mathbb{C}^*)^2$, we have

$$\ln(\mathbf{f}, \gamma) = I_\gamma(f, g),$$

where $\mathbf{f} = (f, g)$.

4.2. Topology and the logarithmic functional. The logarithmic functional has the following topological property. Let M be a real manifold and $\mathbf{f} : M \to (\mathbb{C}^*)^{n+1}$ a smooth map. An n-dimensional cycle $\tilde{\gamma}$ on M can be considered as the image under a piecewise smooth map $\phi : X \to M$ of some n-dimensional cycle γ on some n-dimensional complex X. With the cycle $\tilde{\gamma}$ on M, associate the element $\ln(\mathbf{f} \circ \phi, \gamma)$ of $\mathbb{C}/\mathbb{Z}$. The n-dimensional co-chain thus obtained gives an n-dimensional cohomology class of M with coefficients in the group $\mathbb{C}/\mathbb{Z}$ if and only if $\mathbf{f}^*\omega \equiv 0$.

Theorem. *If the form $\mathbf{f}^*\omega$ is identically equal to zero on M, then the functional that takes a cycle $\phi : \gamma \to M$ to the element $\ln(\mathbf{f} \circ \phi, \gamma) \in \mathbb{C}/\mathbb{Z}$ is a cohomology class in $H^n(M, \mathbb{C}/\mathbb{Z})$.*

Main Example. Let X be an n-dimensional complex algebraic variety, $f_1, \dots, f_{n+1}$ a sequence of $(n+1)$ rational functions on X, and $S(X)$ the singular locus of X. Consider the manifold $M = X \setminus ((f_1) \cup \dots \cup (f_{n+1}) \cup S(X))$ and the map $\mathbf{f} : M \to (\mathbb{C}^*)^{n+1}, \mathbf{f} = (f_1, \dots, f_{n+1})$. Then $\mathbf{f}^*\omega \equiv 0$. So, by the theorem, $\mathbf{f}$ gives rise to a cohomology class in $H^n(M, \mathbb{C}/\mathbb{Z})$. Denote by $\{\ln(\mathbf{f}, \gamma)\}$ its value on a cycle γ.

4.3. Multiplicative version of the logarithmic functional. One can define a multiplicative version of the logarithmic functional that takes a map $\mathbf{f}$ and a cycle γ to the number $\exp(2\pi i \ln(\mathbf{f}, \gamma))$. By the construction, the functional $\exp(2\pi i \ln(\mathbf{f}, \gamma))$ takes values in $\mathbb{C}^*$ and has the following properties:

1) it depends skew-symmetrically on the components of $\mathbf{f}$, for example

$$\exp(2\pi i \ln(f_1, f_2, \dots, f_{n+1}, \gamma)) = (\exp(2\pi i \ln(f_2, f_1, \dots, f_{n+1}, \gamma)))^{(-1)};$$

2) it depends multiplicatively on the components of $\mathbf{f}$, for example

$$\exp(2\pi i \ln(\varphi_1\psi_1, f_2, \dots, f_{n+1}, \gamma)) =$$
$$= \exp(2\pi i \ln(\varphi_1, f_2, \dots, f_{n+1}, \gamma) \exp 2\pi i \ln(\psi_1, f_2, \dots, f_{n+1}, \gamma)).$$

THEOREM. *If the form* $\mathbf{f}^*\omega$ *is identically equal to zero on* M*, then the cochain, whose value on a cycle* γ *equals to* $\exp(2\pi i\{\ln(\mathbf{f},\gamma)\})$*, is a cohomology class in* $H^n(M,\mathbb{C}^*)$.

THEOREM. *Let* $X \subset (\mathbb{C}^*)^n$ *be the real* n*-dimensional torus* $|y_1| = \cdots = |y_n| = 1$ *and* γ *the fundamental cycle of* X*. Consider* $n+1$ *monomials* $c_1\mathbf{y}^{\mathbf{k}_1},\dots,c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}$*. Let* $\mathbf{f}: X \to (\mathbb{C}^*)^{n+1}$ *be the restriction of the map* $(c_1\mathbf{y}^{\mathbf{k}_1},\dots,c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}) : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n+1}$ *to* X*. Then* $\exp(2\pi i\{\ln(\mathbf{f},\gamma)\})$ *is equal to the Parshin symbol* $[c_1\mathbf{y}^{\mathbf{k}_1},\dots,c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}]$.

THEOREM. *Consider* $n+1$ *rational functions* $\mathbf{f} = (f_1,\dots,f_{n+1})$ *on an* n*-dimensional complete complex algebraic variety* X*. Let* F *be a flag on* X *and* $\gamma_F \in H_k(U,\mathbb{Z})$ *the flag-localized homology class. Then* $\exp 2\pi i\{\ln(\mathbf{f},\gamma_F)\}$ *is equal to the Parshin symbol* $[f_1,\dots,f_{n+1}]_F$.

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