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Newton polytopes and irreducible components of complete intersections

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Abstract. We calculate the number of irreducible components of varieties in $(\mathbb{C}^*)^n$ determined by generic systems of equations with given Newton polytopes. Every such component can in its turn be given by a generic system of equations whose Newton polytopes are found explicitly. It is known that many discrete invariants of a variety can be found in terms of the Newton polytopes. Our results enable one to calculate such invariants for each irreducible component of the variety.

Keywords: Newton polytopes, mixed volume, irreducible components, holomorphic forms.

To the memory of Andrey Andreevich Bolibrukh

§ 1. Introduction

1.1. Prehistory. Being a deep mathematician renowned by the whole mathematical community for his solution of Hilbert's 21st problem, Andrey Andreevich Bolibrukh was also a ready communicator with a profound knowledge of human nature, a benevolent and unusually honest person. He willingly helped many people.

Andrey also played a significant role in my life. Let me explain this in more detail. In the early 1970s, when a pupil of Vladimir Igorevich Arnold, I had constructed a new version of Galois theory, which formed the basis of my PhD thesis. I was told to publish this theory in 'Izvestiya of the Academy of Sciences of the USSR'. In those days I knew neither ordinary nor differential Galois theory and was absolutely ignorant of the intricate history of the subject. Therefore I decided to learn a little before writing the paper. However, life goes on. Soon I obtained new results in the theory of Newton polytopes and then developed my theory of fewnomials. Being preoccupied with all this, I forgot about topological Galois theory. The related plans were deferred *sine die*.

Andrey became a co-editor of the new 'Journal of Dynamical and Control Systems', whose first issue was to appear in January 1995. In February 1994 (21 years since the defence of my PhD thesis) Andrey briefly visited us in Toronto. He thought about the journal and remembered my theory in this connection. Andrey

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persuaded me to separate the question of its connections with known mathematics and concentrate on arranging the text. His idea of publishing the paper in the first issue seemed absolutely unrealistic. We were in Canada, and the text was left in our Moscow flat. Somebody had to reach and edit it, translate it into English, and prepare it for publication. Back in Moscow, Andrey got hold of the text, made a photocopy and sent it to us. He infected me with his enthusiasm. I restored all the details and edited the text, our old friend Smilka Zdrawkowska translated it into English, and the paper did indeed appear in the first issue.[1]

As a result, I returned to topological Galois theory. In the thesis, it was developed only for functions of one variable. I was able to construct a multi-dimensional version, make clear the connections with the algebraic and differential Galois theories, and learn the history of the question. A book on this theme was published in 2008.[2] An expanded English version appeared recently.[3]

Nothing of this would probably have happened without Andrey's influence. My paper would have remained unpublished and then completely forgotten.

In connection with this story I would like to dedicate to A. A. Bolibrukh one of my old unpublished results, of which I am proud. Probably the oldest of such results is an affine version of the Bernstein–Kushnirenko theorem. It counts the number and multiplicities of isolated roots in $\mathbb{C}^n$ of a generic system of n polynomial equations with given Newton polytopes (the original theorem counts the number of roots of such a system in $(\mathbb{C}^*)^n$). There had been a number of papers on this theme, all without definitive results. While preparing the affine version for publication, I had found a generalization of it, which counts the number and multiplicities of $(n-k)$-dimensional components of a variety determined in $\mathbb{C}^n$ by a generic system of k polynomial equations with given Newton polytopes. The first thing required for this generalization is to find the number of irreducible components of such a variety in $(\mathbb{C}^*)^n$. When a thorough computation of this number was written down, it became clear that this was a natural place to stop.

1.2. Content of the paper. We calculate the number of irreducible components of a variety determined in $(\mathbb{C}^*)^n$ by a generic system of k equations with given Newton polytopes. When $k = n$ our result coincides with the Bernstein–Kushnirenko theorem. When $k < n$ its proof generalizes the version of the proof of this theorem in [1]. The more complicated affine version of the problem will be treated elsewhere.

Let $(\mathbb{C}^*)^n$ be a complex torus with fixed coordinates $z_1, \dots, z_n$, and let $\mathbb{R}^n$ be the real *character space* with coordinates $x_1, \dots, x_n$. Each integer point $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ determines a character of the torus (or monomial) $z^k = z_1^{k_1} \cdots z_n^{k_n}$.

A *Laurent polynomial* is a finite linear combination $P = \sum c_k z^k$ of characters. The *support* $\operatorname{supp}(P)$ of a Laurent polynomial P is a finite set $\operatorname{supp}(P) \subset \mathbb{Z}^n \subset \mathbb{R}^n$ defined by the condition $k \in \operatorname{supp}(P) \iff c_k \neq 0$. The *Newton polytope* $\Delta(P) \subset \mathbb{R}^n$ of a Laurent polynomial P is the convex hull of $\operatorname{supp}(P)$.

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[2] A. G. Khovanskii, *Topological Galois theory*, MCCME, Moscow 2008. (Russian)

[3] A. Khovanskii, *Topological Galois theory. Solvability and unsolvability of equations in finite terms*, Springer Monographs in Math., Springer, Berlin 2014.

In this paper we deal with an algebraic variety X defined in $(\mathbb{C}^*)^n$ by a system of equations

$$P_1 = \cdots = P_k = 0, \tag{1}$$

where $P_1, \dots, P_k$ is a sufficiently general tuple of Laurent polynomials with given supports $A_1, \dots, A_k \subset \mathbb{Z}^n$ and Newton polytopes $\Delta_1, \dots, \Delta_k \subset \mathbb{R}^n$. Such an algebraic variety is the main object of study in the theory of Newton polytopes. It is well known (see [1], [2]) that X is a non-singular variety and the discrete invariants of X depend only on the tuple of polytopes $\Delta_1, \dots, \Delta_k$ (so they are independent of both the tuple of the supports $A_1, \dots, A_n$ whose convex hulls are equal to $\Delta_1, \dots, \Delta_k$, and the choice of a sufficiently general system of Laurent polynomials with these supports). In what follows, when speaking of a generic system of equations (1), we shall sometimes omit any mention of the set of supports $A_1, \dots, A_k$ and indicate only the Newton polytopes $\Delta_1, \dots, \Delta_k$. In this case we always have in mind a certain tuple of supports and do not mention it explicitly only to avoid a flood of easily recoverable details.

A common assumption in the study of (1) is that all the polytopes Δ_i have full dimension n. We do not make this assumption. For us, an important role is played by the notion of the *defect* of a set of indices $J \subset \{1, \dots, k\}$ labelling a tuple of convex bodies $\Delta_1, \dots, \Delta_k$. For every non-empty J we define a body $\Delta_J = \sum_{i \in J} \Delta_i$ (the sum is understood in the *Minkowski sense*; see § 2.1). We denote the number of elements in J by $|J|$. When $J = \varnothing$, we put $\Delta_J = \{0\}$ and $|J| = 0$.

Definition. The *defect* of a set $J \subset \{1, \dots, k\}$ for a tuple $\Delta_1, \dots, \Delta_k$ is the number $\dim \Delta_J - |J|$. In particular, the defect of the empty set is equal to zero. A tuple of bodies is said to be *independent* if the defect of every set J for this tuple is non-negative.

Here are some results related to the notion of defect.

Theorem (Minkowski). *A tuple of bodies $\Delta_1, \dots, \Delta_n$ has mixed volume zero if and only if it is dependent.*

The notion of mixed volume (see § 2.1) and Minkowski's theorem are important in the present paper. A detailed proof of this theorem is given in § 2.2. It follows from Minkowski's theorem combined with the Bernstein–Kushnirenko theorem (see § 3) that *a sufficiently general system* (1) *is incompatible if and only if the polytopes $\Delta_1, \dots, \Delta_k$ are dependent* (see § 3, Theorem 11).

By the reduction theorem (§ 5.1, Theorem 15), when a generic system (1) is compatible but *there is a set of indices J with zero defect, the system can be reduced to some number of generic systems of equations in a smaller number of variables* (the number of systems is determined by J and the tuple of Newton polytopes). By the irreducibility theorem (§ 5.2, Theorem 17), *if such a reduction is impossible, then the variety given by the system* (1) *is irreducible.*

The reduction theorem and the irreducibility theorem enable us to find the number of irreducible components of X and compute their discrete invariants. The computation of discrete invariants (as well as a major part of the results in the theory of Newton polytopes) is based on a connection (discovered in [1], [2]) between this theory and the geometry of toric varieties. However, there are elementary results for which this technique is unnecessary. We start the paper with these results. One of them says that X is smooth. We complement it by proving that X is transversal to any given subvariety (see § 1.3).

Let $h^p(M)$ be the dimension of the space of holomorphic p-forms on a smooth compact complex algebraic variety M. This number is a birational invariant of M. Hence for all (non-compact smooth) algebraic varieties Y one can define $h^p(Y)$ to be the number $h^p(M)$, where M is any smooth compact variety birationally equivalent to Y. The alternating sum $\chi(Y) = \sum(-1)^p h^p(Y)$ of the numbers $h^p(Y)$ is called the *arithmetic genus* of an algebraic variety Y.

There is an explicit formula for the arithmetic genus $\chi(X)$ of a variety X determined by a sufficiently general system (1) (see Theorem 29 and subsequent remarks in § 8). We complement this formula by the following results.

1) If a number $j > 0$ *is not equal to the defect* of a non-empty set of indices J for an independent tuple of Newton polytopes of the system (1), *then* $h^j(X)$ *is equal to zero* (§ 8, Theorem 31).

2) *But if* j *is equal to the defect* of some non-empty set of indices J, *then under certain additional assumptions the number* $h^j(X)$ *is positive* (Theorem 20, § 5.2).

The assertion on the positivity of $h^j(X)$ follows from an explicit construction of holomorphic j-forms on the closure of X in an appropriate toric compactification. It uses no cohomology calculations and is completely elementary. The cohomology of toric varieties with coefficients in sheaves of sections of one-dimensional invariant holomorphic bundles was used in the theory of Newton polytopes for the first time in [1], [2]. Our present results might have appeared there (much more refined calculations using the theory of mixed Hodge structures appeared later, starting with [3]).

Before describing the main content of the paper, we show that the variety $X \subset (\mathbb{C}^*)^n$ determined by a sufficiently general system of equations (1) is smooth and transversal to any fixed semi-algebraic set $Y \subset (\mathbb{C}^*)^n$.

1.3. Smoothness and transversality. Let $Y \subset (\mathbb{C}^*)^n$ be a semi-algebraic set. Suppose that Y is stratified, that is, represented as a finite union $Y = \bigcup Y_i$ of disjoint smooth (generally speaking, non-closed) algebraic varieties Y_i.

Theorem 1. *For almost all tuples of coefficients of the system* (1), *the variety* $X \subset (\mathbb{C}^*)^n$ *determined by this system is non-singular and transversal to all strata* Y_i *of the semi-algebraic set* Y.

Proof. Enumerate the monomials in the support of each of the Laurent polynomials $P_i = \sum c_{i,j} x^{m_{i,j}}$ in an arbitrary way, take the first monomial $x^{m_{i,1}}$ and write P_i in the form $P_i = c_{i,1} x^{m_{i,1}} + \widetilde{P}_i$, where $\widetilde{P}_i = \sum_{j>1} c_{i,j} x^{m_{i,j}}$. One can rewrite the system (1) in the form $-c_{1,1} = \widetilde{P}_1 \cdot x^{-m_{1,1}}, \dots, -c_{k,1} = \widetilde{P}_k \cdot x^{-m_{k,1}}$. In other words, X is the pre-image of the point $c = (-c_{1,1}, \dots, -c_{1,k}) \in \mathbb{C}^k$ under the map $(\widetilde{P}_1, \dots, \widetilde{P}_k)\colon (\mathbb{C}^*)^n \to \mathbb{C}^k$. By the Sard–Bertini theorem, X is non-singular for almost all points $c \in \mathbb{C}^k$. Using the same theorem for the restriction of the map to Y_i, we see that X is transversal to Y_i for almost all points $c \in \mathbb{C}^k$. $\square$

Corollary 2. *For every semi-algebraic set $Y \subset (\mathbb{C}^*)^n$ and every variety X determined by a system* (1) *with sufficiently general coefficients, the set $Y \cap X$ is dense in X if and only if Y is dense in $(\mathbb{C}^*)^n$.*

Proof. It is known that every semi-algebraic set can be stratified. Hence Corollary 2 follows from Theorem 1. $\square$

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§ 2. Mixed volume and Minkowski's theorem

2.1. Mixed volume. One of the first results in the theory of Newton polytopes is the Bernstein–Kushnirenko theorem. Below we recall the statement of this famous theorem. To do this, we need some notions from convex geometry.

A subset A of a real vector space M can be multiplied by $\lambda \in \mathbb{R}$: the set λA is defined by putting

$$c \in \lambda A \quad \iff \quad \exists\, a \in A \mid c = \lambda a.$$

Subsets $A, B \subset M$ can be added: the set $A + B$ is defined by putting

$$c \in A + B \quad \iff \quad \exists\, a \in A,\ \exists\, b \in B \mid c = a + b$$

and is called the *Minkowski sum* of A and B. If A and B are *convex bodies* (that is, convex bounded closed sets), then so are the sets λA and $A + B$.

We endow an n-dimensional space M with a fixed translation-invariant Lebesgue measure μ (such a measure is unique up to a positive multiplier). The *volume* $V(\Delta)$ *of a convex body* $\Delta \subset M$ is its measure $\mu(\Delta)$.

Definition. The *mixed volume* of n convex bodies $\Delta_1, \dots, \Delta_n$ is a number $V(\Delta_1, \dots, \Delta_n)$ defined by the following *polarization formula*:

$$n!\, V(\Delta_1, \dots, \Delta_n) = (-1)^{n-1} \sum_{1 \leqslant i \leqslant n} V(\Delta_i) + (-1)^{n-2} \sum_{1 \leqslant i_1 < i_2 \leqslant n} V(\Delta_{i_1} + \Delta_{i_2}) + \dots + V(\Delta_1 + \dots + \Delta_n).$$

The notion of mixed volume was introduced by Minkowski, who showed that it is the unique function of n convex bodies which is

1) symmetric (that is, invariant under permutations of the arguments);

2) multilinear (linearity with respect to the first argument means that

$$V(\lambda_1 \Delta_1' + \lambda_2 \Delta_1'', \dots, \Delta_n) = \lambda_1 V(\Delta_1', \dots, \Delta_n) + \lambda_2 V(\Delta_1'', \dots, \Delta_n)$$

for all $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$);

3) equal to the volume on the diagonal, that is, $V(\Delta, \dots, \Delta) = \Delta$.

We easily see that the mixed volume is *monotone*: if $\Delta_1' \subset \Delta_1, \dots, \Delta_n' \subset \Delta_n$, then $V(\Delta_1', \dots, \Delta_n') \leqslant V(\Delta_1, \dots, \Delta_n)$. It is also *non-negative*. This follows from monotonicity and the equality $V(\Delta_1', \dots, \Delta_n') = 0$, where the $\Delta_i' \in \Delta_i$ are points.

Lemma 3. *Let $I_1, \dots, I_n$ be closed intervals and let $\Pi = I_1 + \dots + I_n$ be the parallelepiped equal to their sum. Then*

$$V(I_1, \dots, I_n) = \frac{1}{n!} \operatorname{Vol}(\Pi). \tag{2}$$

Proof. The equality (2) follows immediately from the polarization formula for the mixed volume. □

In § 2.4 we prove Minkowski's theorem (see § 1.2), which gives a criterion for the vanishing of the mixed volume.

2.2. A criterion for the vanishing of the mixed volume. The proof of the criterion uses simple linear algebra.

Definition. A tuple of affine subspaces $L_1, \dots, L_m$ in a vector space M is said to be *non-degenerate* if for every non-empty set of indices $J \subset \{1, \dots, m\}$ we have $\dim \sum_{i \in J} L_i \geqslant |J|$, where $|J|$ is the number of elements in J. A tuple $L_1, \dots, L_m$ is said to be *degenerate* otherwise.

For every convex body $\Delta \subset L$ we denote by $L(\Delta)$ the minimal affine subspace that contains Δ. We easily see that *a tuple of convex bodies $\Delta_1, \dots, \Delta_m \subset L$ is dependent* (see § 1.2) *if the tuple of affine subspaces $L(\Delta_1), \dots, L(\Delta_m)$ is degenerate.*

2.3. A theorem of linear algebra. Let $M_1, \dots, M_n$ be vector subspaces of an ambient space M.

Theorem 4. *The tuple of spaces $M_1, \dots, M_n$ is non-degenerate if and only if one can choose independent vectors $\{v_i\}$, $1 \leqslant i \leqslant n$, in $M_1, \dots, M_n$ in such a way that $v_1 \in M_1, \dots, v_n \in M_n$.*

Before proving Theorem 4, we establish Lemmas 5–7 (of which only Lemma 6 is not completely obvious).

Lemma 5. *If there are independent vectors $v_1 \in M_1, \dots, v_n \in M_n$, then the tuple of spaces $M_1, \dots, M_n$ is non-degenerate.*

Proof. For every subtuple $M_{i_1}, \dots, M_{i_k}$ we have $\dim(M_{i_1} + \dots + M_{i_k}) \geqslant k$ since the vectors $v_{i_1} \in M_{i_1}, \dots, v_{i_k} \in M_{i_k}$ are linearly independent. □

Lemma 6. *If the tuple of spaces $M_1, \dots, M_n$ is non-degenerate, then there is a space $V_n \subset M_n$ such that $\dim V_n = 1$ and the tuple of spaces $M_1, \dots, M_{n-1}, V_n$ is non-degenerate.*

Proof. By hypothesis, for every non-empty $J \subset \{1, \dots, n\}$ we have $\dim M_J \geqslant |J|$. Let J^* be a non-empty subset of $\{1, \dots, n-1\}$. We say that J^* is *of the first type* if $M_{J^*} \cap M_n \neq M_n$. But if $M_{J^*} \cap M_n = M_n$, then we say that J^* is *of the second type*. If J^* is of the second type, then $\dim M_{J^*} \geqslant (|J^*| + 1)$. Indeed, if $M_n \subset M_{J^*}$, then $M_{J^*} \cup M_n = M_{J^*}$. But $M_{J^*} \cup M_n = M_J$, where $J = J^* \cup \{n\}$ and, by hypothesis, $\dim M_J$ is not less than $|J| = |J^*| + 1$.

The union D of all sets of the form $M_{J^*} \cap M_n$, where J^* is a set of the first type, cannot be equal to M_n (the union of finitely many proper subspaces cannot cover the whole space). We claim that the linear span of any non-zero vector $v \in M_n \setminus D$ can be taken as V_n.

Indeed, let us show that the tuple of spaces $T_1, \dots, T_n$ is independent, where $T_i = M_i$ for $1 \leqslant i \leqslant n-1$ and $T_n = V_n$ is spanned by a non-zero vector $v \in M_n \setminus D$. For every non-empty set $J^* \subset \{1, \dots, n-1\}$ we have $T_{J^*} = M_{J^*}$ and $\dim T_{J^*} = \dim M_{J^*} \geqslant |J^*|$. Consider the set $J = J^* \cup \{n\}$. If J^* is of the first type, then the space $T_n = V_n$ is not contained in $T_{J^*} = M_{J^*}$ and $\dim T_J \geqslant \dim T_{J^*} + 1 \geqslant |J^*| + 1 = |J|$. If J^* is of the second type, then $T_n \subset T_{J^*}$ and $\dim T_J = \dim T_{J^*} = \dim M_{J^*} \geqslant |J^*| + 1 = |J|$. $\square$

Lemma 7. *If the tuple of spaces $M_1, \dots, M_n$ is non-degenerate, then there is a non-degenerate tuple of one-dimensional spaces $V_1, \dots, V_n$ such that $V_1 \subset M_1, \dots, V_n \subset M_n$.*

Proof. By Lemma 6, the space M_n in the tuple $M_1, \dots, M_n$ can be replaced by a one-dimensional space V_n in such a way that the tuple $M_1, \dots, M_{n-1}, V_n$ remains independent. Relabelling the spaces and using Lemma 6, one can replace M_{n-1} by a one-dimensional subspace V_{n-1} in such a way that the tuple $M_1, \dots, M_{n-2}, V_{n-1}, V_n$ remains independent, and so on. $\square$

Theorem 4 follows from Lemmas 5 and 7.

2.4. Minkowski's theorem. We use Theorem 4 to prove Minkowski's theorem, which will be split into Theorems 8 and 9.

Theorem 8. *If the tuple of affine spaces $L(\Delta_1), \dots, L(\Delta_n)$ is non-degenerate, then $V(\Delta_1, \dots, \Delta_n) > 0$.*

Proof. Let $M_1, \dots, M_n$ be vector spaces parallel to the affine spaces $L(\Delta_1), \dots, L(\Delta_n)$. Since the tuple $M_1, \dots, M_n$ is independent, Theorem 4 enables us to choose independent vectors $v_1 \in M_1, \dots, v_n \in M_n$. Let $I_1 \subset \Delta_1, \dots, I_n \subset \Delta_n$ be closed intervals parallel to the lines spanned by the vectors $v_1, \dots, v_n$. Then $V(I_1, \dots, I_n) > 0$ by (2). Since the mixed volume is monotone, we have $V(\Delta_1, \dots, \Delta_n) \geqslant V(I_1, \dots, I_n) > 0$. $\square$

Theorem 9. *Suppose that the tuple of affine subspaces $L(\Delta_1), \dots, L(\Delta_n)$ is degenerate. Then $V(\Delta_1, \dots, \Delta_n) = 0$.*

We first prove the following lemma.

Lemma 10. *Theorem 9 holds if we also know that each of the bodies $\Delta_1, \dots, \Delta_n$ is a parallelepiped.*

Proof. The mixed volume is translation-invariant. Therefore we can assume that the point 0 is a vertex of each of the parallelepipeds Δ_i. Let $I_i^1, \dots, I_i^{m_i}$ be the edges incident to 0 in the parallelepiped Δ_i, where $i = 1, \dots, n$. For every i we have $\Delta_i = I_i^1 + \dots + I_i^{m_i}$. Since the mixed volume is multilinear,

$$V(\Delta_1, \dots, \Delta_n) = \sum_{1 \leqslant j_1 \leqslant m_1, \dots, 1 \leqslant j_n \leqslant m_n} V(I_1^{j_1}, \dots, I_n^{j_n}).$$

By Theorem 4 and formula (2) we have $V(I_1^{j_1}, \dots, I_n^{j_n}) = 0$. $\square$

Proof of Theorem 9. For each of the bodies Δ_i we choose an arbitrary parallelepiped Δ'_i which contains Δ_i and lies in $L(\Delta_i)$. Then $V(\Delta'_1,\dots,\Delta'_n) = 0$ by Lemma 10. Since the mixed volume is monotone and non-negative, we have $V(\Delta'_1,\dots,\Delta'_n) \geqslant V(\Delta_1,\dots,\Delta_n) \geqslant 0$. This completes the proof of Theorem 9 and, therefore, of Minkowski's theorem. $\square$

§ 3. The Bernstein–Kushnirenko theorem and related results

If we fix a discrete lattice Λ of full rank in an n-dimensional vector space M, then M becomes endowed with a translation-invariant *integer volume*.

Definition. An *integer volume* on a space $M \supset \Lambda$ is the volume associated with a translation-invariant measure μ normalized by the condition $\mu(\Pi) = 1$, where Π is the parallelepiped spanned by some vectors $e_1,\dots,e_n$ that generate Λ.

In this section we consider an integer volume on the space $\mathbb{R}^n$ containing the lattice $\mathbb{Z}^n$ in which the supports of Laurent polynomials lie.

What is the number of roots in $(\mathbb{C}^*)^n$ of a generic system (1) of $k = n$ equations with supports $A_1,\dots,A_n \subset \mathbb{Z}^n$ whose convex hulls are equal to $\Delta_1,\dots,\Delta_n$? The answer is given by the following famous theorem.

Theorem (Bernstein–Kushnirenko). *A generic system* (1) *of $k = n$ equations has only non-multiple roots in* $(\mathbb{C}^*)^n$, *and their number is equal to* $n!\,V(\Delta_1,\dots,\Delta_n)$.

To date there are many proofs of this remarkable theorem. The scheme of one of them is given in § 8 after Theorem 29. Bernstein found a criterion for the compatibility of a generic system (1).

Theorem 11. *A sufficiently general system* (1) *is incompatible if and only if the polytopes $\Delta_1,\dots,\Delta_k$ are dependent.*

Proof. It is easy to prove that a sufficiently general system (1) either determines a smooth $(n-k)$-dimensional variety or is inconsistent. Therefore if $k > n$, then (1) is incompatible. The polytopes $\Delta_1,\dots,\Delta_k \subset \mathbb{R}^n$ are automatically dependent when $k > n$, and this completes the proof for $k > n$. When $k = n$, the theorem follows immediately from the Bernstein–Kushnirenko theorem and Minkowski's theorem. Suppose that $k < n$. Then we complement the system (1) by generic linear equations

$$P_{n-k+1} = \dots = P_n = 0 \tag{3}$$

of the form $P_i = \sum a_{i,j} z_j + b_i = 0$, where z_j are the coordinates on the torus $(\mathbb{C}^*)^n$ and $a_{i,j}$, b_i are sufficiently general complex numbers. The Newton polytopes $\Delta_{n-k+1},\dots,\Delta_n$ of the linear polynomials $P_{n-k+1},\dots,P_n$ are all equal to the standard n-dimensional unit simplex Δ. A sufficiently general system formed by the equations (1) and (3) is compatible if and only if the set of solutions of (1) is non-empty. This again reduces the case $k < n$ to the Bernstein–Kushnirenko theorem and Minkowski's theorem since the tuples of polytopes $\Delta_1,\dots,\Delta_k$ and $\Delta_1,\dots,\Delta_k,\Delta_{n-k+1},\dots,\Delta_n$ are either both dependent or both independent. $\square$

We restate Theorem 11.

Theorem 11′. *A sufficiently general system* (1) *is incompatible if and only if the defect of some set J for the tuple of polytopes $\Delta_1, \dots, \Delta_k$ is negative.*

How to describe in geometric terms those tuples of supports $A_1, \dots, A_n \subset \mathbb{Z}^n$ for which a generic system (1) of $k = n$ equations has exactly one root? In other words, we ask for a description of those tuples of integer polytopes $\Delta_1, \dots, \Delta_n$ whose mixed volume is equal to $1/n!$. Theorem 12, which was obtained in [5], answers this question by reducing it to the corresponding question in a lower-dimensional space. To state Theorem 12, we need some general definitions that will be used throughout.

The Newton polytopes Δ_i lie in an n-dimensional vector space M endowed with a lattice $\Lambda \approx \mathbb{Z}^n$. For each non-empty set of indices J let $L(\Delta_J)$ be the affine space spanned by the polytope Δ_J and let $M(\Delta_J)$ be the vector space parallel to $L(\Delta_J)$. We denote the quotient space of M with respect to $M(\Delta_J)$ by $M^\perp(\Delta_J)$.

The space $M(\Delta_J)$ contains a lattice $\Lambda_J = \Lambda \cap M(\Delta_J)$ of full rank. Hence an integer volume is defined in $M(\Delta_J)$. Its polarization (the integer mixed volume *in the sense of* $M(\Delta_J)$) is defined for all $\dim M(\Delta_J)$-tuples of polytopes Δ_i such that every space $L(\Delta_i)$ can be moved to $M(\Delta_J)$ by a shift. (This notion of mixed volume is used in the statements of Theorems 15 and 19 below.)

The space $M^\perp(\Delta_J)$ contains the quotient lattice $\Lambda^\perp = \Lambda/\Lambda_J$ of full rank and the $(k-|J|)$-tuple of polytopes $\{\pi_{J^\perp}\Delta_i\}$, where $i \in J^\perp = \{1, \dots, k\} \setminus J$. By definition, the polytope $\pi_{J^\perp}\Delta_i \subset M^\perp(\Delta_J)$ is the image of the polytope $\Delta_i \subset M(\Delta_J)$ under the factorization map $M(\Delta_J) \to M^\perp(\Delta_J)$. The polytope $\pi_{J^\perp}\Delta_i \subset M^\perp(\Delta_J)$ is integer, that is, its vertices lie in the lattice $\Lambda^\perp$.

Here is another definition. An integer k-dimensional simplex Δ with vertices $T_0, \dots, T_k$ is said to be *primitive* if the vectors $e_i = T_i - T_0$, $i = 1, \dots, k$, generate the lattice $\Lambda \cap M(\Delta)$. The integer volume of a primitive simplex is equal to $1/k!$.

Theorem 12 (see [5]). *Integer polytopes $\Delta_1, \dots, \Delta_n$ have integer mixed volume $1/n!$ if and only if the following conditions hold.*

1) *The polytopes in the tuple are independent.*

2) *There is a subtuple of $k > 0$ polytopes which lie* (*up to a shift*) *in some primitive k-dimensional simplex Δ such that the images of the other $n-k$ polytopes in $M/M(\Delta)$ have integer mixed volume $1/(n-k)!$.*

§ 4. Auxiliary results

4.1. Decreasing the number of unknowns. Let the Newton polytopes $\Delta_1, \dots, \Delta_k$ of the Laurent polynomials in (1) be such that the polytope $\Delta = \sum_{1 \leqslant i \leqslant k} \Delta_i$ has dimension $m < n$. Then the system (1) can be reduced to a system with m unknowns. We recall how to do this.

Let $M(\Delta)$ be the vector space parallel to the affine space $L(\Delta)$ that contains the polytope Δ. Then $M(\Delta)$ has dimension m and contains a lattice Λ_Δ of full rank. The lattice Λ_Δ can be identified with the character lattice $\mathbb{Z}^m$ of the m-dimensional torus $(\mathbb{C}^*)^m$. The embedding $\Lambda_\Delta \subset \Lambda \approx \mathbb{Z}^n$ is dual to the homomorphism $\pi_\Delta \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ of the torus $(\mathbb{C}^*)^n$ onto the torus $(\mathbb{C}^*)^m$. Note that $(\mathbb{C}^*)^n$ can be written as a direct product $(\mathbb{C}^*)^n = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n-m}$ in such a way that the map π_Δ coincides with the projection π_1 of $(\mathbb{C}^*)^n$ to the first factor.

Multiplying the Laurent polynomial P_i by an appropriate monomial z^{q_i}, we can ensure that the support $A_i + q_i$ of the Laurent polynomial $z^{q_i}P_i$ lies in $M(\Delta)$. The variety X remains unchanged because the equations $P_i = 0$ and $z^{q_i}P_i = 0$ are equivalent to each other on the torus $(\mathbb{C}^*)^n$. Since Λ_Δ is isomorphic to the character lattice $\mathbb{Z}^m$ of the torus $(\mathbb{C}^*)^m$, we can assume that the supports $B_i = A_i + q_i$ lie in $\mathbb{Z}^m$ and the Laurent polynomial $z^{q_i}P_i$ may be regarded as a Laurent polynomial Q_i on $(\mathbb{C}^*)^m$. Moreover, we have $z^{q_i}P_i = \pi_\Delta^* Q_i$.

Thus the system (1) on $(\mathbb{C}^*)^n$ induces a system of equations

$$Q_1 = \dots = Q_k = 0 \tag{4}$$

on $(\mathbb{C}^*)^m$. This system depends on the smaller number $m < n$ of variables. The relation between the variety $\widetilde{X}$ determined by (4) and the variety X determined by (1) is given by the formula $X = \pi_\Delta^{-1}(\widetilde{X})$. Since π_Δ is equivalent to the projection π_1 of the torus $(\mathbb{C}^*)^n = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n-m}$ onto the first factor, we obtain that X is isomorphic to $\widetilde{X} \times (\mathbb{C}^*)^{n-m}$. Thus the study of X reduces to a study of the variety $\widetilde{X}$ determined by a system depending on a smaller number of variables.

4.2. Varieties determined by the system (1) and its subsystems. Let $J = \{i_1, \dots, i_l\}$ and $J^\perp = \{j_1, \dots, j_{k-l}\}$ be non-empty complementary subsets of the set $\{1, \dots, k\}$ indexing the equations in (1). We consider two systems of the form (1) on $(\mathbb{C}^*)^n$:

$$P_{i_1} = \dots = P_{i_l} = 0, \tag{5}$$

$$P_{j_1} = \dots = P_{j_{k-l}} = 0. \tag{6}$$

Suppose that the polytope $\Delta_J = \sum_{i \in J} \Delta_i$ has dimension $m < n$. There is no loss of generality in assuming that the Newton polytopes $\Delta_{i_1}, \dots, \Delta_{i_l}$ lie in the vector space $M(\Delta_J)$. The system (5) is related to a system of equations

$$Q_{i_1} = \dots = Q_{i_l} = 0 \tag{7}$$

on $(\mathbb{C}^*)^m$ such that $P_i = \pi_{\Delta_J}^* Q_i$ for $i \in J$ and $\pi_{\Delta_J} \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ is the homomorphism described in the previous subsection.

Let X_1, X_2 and $X = X_1 \cap X_2$ be the varieties in $(\mathbb{C}^*)^n$ determined by (5), (6), and the system of the form (1) containing all equations in (5) and (6). Let $\widetilde{X}_1$ be the variety in $(\mathbb{C}^*)^m$ determined by the system (7). Which tuples $\Delta_1, \dots, \Delta_k$ of Newton polytopes are such that the image $\pi_{\Delta_J}(X)$ of X under the homomorphism $\pi_{\Delta_J} \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ is dense in $\widetilde{X}_1$? The following theorem gives an answer.

Theorem 13. *The image $\pi_{\Delta_J}(X)$ of X under the homomorphism*

$$\pi_{\Delta_J} \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$$

is dense in $\widetilde{X}_1$ if and only if the defect of J for the tuple $\Delta_1, \dots, \Delta_k$ is not greater than the defect of any set of indices $\tilde{J}$ containing J.

We postpone the proof until the end of this subsection.

Let us now find conditions under which the image of a full intersection under a homomorphism is dense in the quotient torus.

Consider a homomorphism $\pi \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ of $(\mathbb{C}^*)^n$ onto $(\mathbb{C}^*)^m$. Let $X \subset (\mathbb{C}^*)^n$ be the algebraic variety determined by a generic system of equations (1).

We consider the restriction $\pi\colon X \to (\mathbb{C}^*)^m$ of π to X and pose the following question. Which tuples $A_1, \dots, A_k$ of supports are such that the image $\pi(X)$ is dense in $(\mathbb{C}^*)^m$? The answer depends on the homomorphism π and on the convex hulls $\Delta_1, \dots, \Delta_k$ of the supports $A_1, \dots, A_k$.

One might think that since the system (1) is generic, the density of the image is guaranteed already by the assumption that $n-k \geqslant m$. If the polytopes $\Delta_1, \dots, \Delta_k$ have full dimension n, then this is indeed the case. But the general answer is more complicated.

The homomorphism π induces an embedding $\pi^*\colon \mathbb{R}^m \to \mathbb{R}^n$ of the vector space $\mathbb{R}^m$ spanned by the character lattice of $(\mathbb{C}^*)^m$ in the vector space $\mathbb{R}^n$ spanned by the character lattice of $(\mathbb{C}^*)^n$.

For brevity we put $M = \pi^*(\mathbb{R}^m)$ and denote the factorization map $\mathbb{R}^n \to \mathbb{R}^n/M = M^\perp$ by $\pi_{M^\perp}$.

Theorem 14. *The image $\pi(X)$ is dense in $(\mathbb{C}^*)^m$ if and only if the polytopes $\pi_{M^\perp}(\Delta_1), \dots, \pi_{M^\perp}(\Delta_k) \subset M^\perp$ are independent.*

Proof. Consider the system of equations $Q_1 = \dots = Q_m = 0$ on the torus $(\mathbb{C}^*)^m$, where $Q_1, \dots, Q_m$ is a generic tuple of linear polynomials in the coordinates $z_1, \dots, z_m$ of $(\mathbb{C}^*)^m$. The polynomials $Q_1, \dots, Q_m$ have the same Newton polytope: the standard unit simplex $\Delta \subset \mathbb{R}^m$, which is defined by saying that $x \in \Delta$ if and only if all coordinates of x are non-negative and $\sum x_i \leqslant 1$. Let $z \in (\mathbb{C}^*)^m$ be a solution of this system. The image $\pi(X)$ contains z if and only if the system $P_1 = \dots = P_k = \pi^*(Q_1) = \dots = \pi^*(Q_m) = 0$ is compatible on $(\mathbb{C}^*)^n$. The Newton polytopes of the Laurent polynomials $\pi^*(Q_j)$ are equal to $\pi^*(\Delta)$. Such a system is compatible if and only if the polytopes $\Delta_1, \dots, \Delta_k$, $\pi^*(\Delta), \dots, \pi^*(\Delta)$ are independent in $\mathbb{R}^n$. This condition is equivalent to the independence of the polytopes $\pi_{M^\perp}(\Delta_1), \dots, \pi_{M^\perp}(\Delta_k)$ in $M^\perp$. □

Proof of Theorem 13. We apply Theorem 14 to the variety $X \subset (\mathbb{C}^*)^n$ and the homomorphism π_{Δ_J} that occur in this theorem. The image of the variety $X = X_1 \cap X_2$ under the homomorphism π_{Δ_J} lies in $\widetilde{X}_1$. Which tuples $\Delta_1, \dots, \Delta_k$ of polytopes are such that $\pi_{\Delta_J}(X)$ is dense in $\widetilde{X}_1$? Here is the answer: $\pi_{\Delta_J}(X)$ *is dense in* $\widetilde{X}_1$ *if and only if* $\pi_{\Delta_J}(X_2)$ *is dense in* $(\mathbb{C}^*)^m$. Indeed, $\pi_{\Delta_J}(X) = \pi_{\Delta_J}(X_1 \cap X_2) = \widetilde{X}_1 \cap \pi_{\Delta_J}(X_2)$ (the last equality holds because $X_1 = \pi^{-1}_{\Delta_J}(\widetilde{X}_1)$). Hence the desired assertion follows from Corollary 2, where the role of the semi-algebraic set Y is played by $\pi_{\Delta_J}(X_2)$, the role of the system (1) by (7), and the role of X by $\widetilde{X}_1$. Theorem 14 yields that *the set* $\pi_{\Delta_J}(X_2) \cap \widetilde{X}_1$ *is dense in* $\widetilde{X}_1$ *if and only if the tuple of polytopes* $\{\pi_{\Delta_J}(\Delta_j)\}$ *for* $j \in J^\perp$ *is independent in* $M^\perp(\Delta_J)$.

It remains to restate the answer. The independence of the tuple of polytopes $\{\pi_{\Delta_J}(\Delta_j)\}$ for $j \in J^\perp$ means that $\dim \pi_{\Delta_J}(\Delta_{J^*}) - |J^*| \geqslant 0$ for every subset $J^* \subset J^\perp$. Put $\tilde{J} = J \cup J^*$. We have

$$\dim \Delta_{\tilde{J}} = \dim(\Delta_J) + \dim \pi_{\Delta_J}(\Delta_{J^*}), \qquad |\tilde{J}| = |J| + |J^*|,$$

whence

$$|\dim \Delta_{\tilde{J}}| - |\tilde{J}| = (|\dim \Delta_J| - |J|) + (|\dim \Delta_{J^*}| - |J^*|) \geqslant |\dim \Delta_J| - |J|. \quad \square$$

§ 5. Main results

5.1. The reduction theorem. We claim that if the generic system (1) is compatible and admits a set of indices J with zero defects, then it may be reduced to generic systems in a smaller number of variables.

Consider the algebraic variety $X \subset (\mathbb{C}^*)^n$ determined by a generic system of equations (1) with Newton polytopes $\Delta_1, \dots, \Delta_k$. Suppose that the following conditions hold.

1) The tuple of polytopes $\Delta_1, \dots, \Delta_k$ is independent.

2) The defect of the set $J = \{i_1, \dots, i_m\}$ for this tuple is equal to zero.

Theorem 15 (the reduction theorem). *Under the conditions* 1), 2) *there is a partition of X into $q = m!\, V(\Delta_{i_1}, \dots, \Delta_{i_m})$ disjoint subvarieties X_i, where*:

1) *V is the polarization of the integer volume in $M(\Delta_J)$*;

2) *each subvariety X_i is isomorphic to a variety determined in $(\mathbb{C}^*)^{n-m}$ by a generic system of equations with the tuple of polytopes $\{\pi_{J^\perp}\Delta_i)$ for i ranging over $J^\perp$.*

We first prove the reduction theorem in a special case.

Write the torus $(\mathbb{C}^*)^n$ as a product $(\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n-m}$ of the tori $(\mathbb{C}^*)^m$ and $(\mathbb{C}^*)^{n-m}$ and let $\pi_1 \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$, $\pi_2 \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-m}$ be the projections onto the factors. Let $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ be the corresponding representation of the vector space $\mathbb{R}^n$ of characters of $(\mathbb{C}^*)^n$ as a product of the corresponding spaces $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ for the tori $(\mathbb{C}^*)^m$ and $(\mathbb{C}^*)^{n-m}$, and let $\tilde{\pi}_1$, $\tilde{\pi}_2$ be the projections onto the factors: $\tilde{\pi}_1 \colon \mathbb{R}^n \to \mathbb{R}^m$, $\tilde{\pi}_2 \colon \mathbb{R}^n \to \mathbb{R}^{n-m}$.

Consider an algebraic variety $X \subset (\mathbb{C}^*)^n = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n-m}$ determined by a generic system of equations (1) with Newton polytopes $\Delta_1, \dots, \Delta_k \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. Suppose that the following conditions hold.

1) The tuple of polytopes $\Delta_1, \dots, \Delta_k$ is independent.

2) The defect of the set $J_m = \{1, \dots, m\}$ for this tuple is equal to zero.

3) We have $\Delta_i \subset \mathbb{R}^m \times \{0\}$ for $1 \leqslant i \leqslant m$, where $0 \in \mathbb{R}^{n-m}$.

Theorem 16. *Under the conditions* 1)–3) *there is a partition of X into $q = m!\, V(\Delta_1, \dots, \Delta_m)$ disjoint subvarieties X_i, where*:

1) *V is the polarization of the integer volume in $\mathbb{R}^m \times 0$*;

2) *each subvariety X_i is isomorphic to a variety determined in $(\mathbb{C}^*)^{n-m}$ by a generic system of equations with polytopes $\tilde{\pi}_2(\Delta_{m+1}), \dots, \tilde{\pi}_2(\Delta_k)$.*

Proof. Since the Newton polytopes $\Delta_1, \dots, \Delta_m$ lie in $\mathbb{R}^m \times \{0\}$, the first m equations $P_1 = \dots = P_m = 0$ of the system (1) may be regarded as equations on the torus $(\mathbb{C}^*)^m \times 1$, where 1 is the identity of the torus $(\mathbb{C}^*)^{n-m}$. By the Bernstein–Kushnirenko theorem, the set $\{(u_i \times 1)\}$ of solutions of this system contains $q = m!\, V(\Delta_1, \dots, \Delta_m)$ elements. The subsystem $P_1 = \dots = P_m = 0$ of (1), regarded as a system of equations on $(\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n-m}$, determines q shifted tori $\{u_i \times (\mathbb{C}^*)^{n-m}\}$, which are biregularly isomorphic to $(\mathbb{C}^*)^{n-m}$. On each of them, the remaining equations $P_{m+1} = \dots = P_k = 0$ of the system (1) determine a subvariety X_i. The isomorphism $u_i \times (\mathbb{C}^*)^{n-m} \to (\mathbb{C}^*)^{n-m}$ transforms the system of equations

$P_{m+1} = \cdots = P_k = 0$ on the shifted torus $u_i \times (\mathbb{C}^*)^{n-m}$ to a sufficiently general system of equations on the torus $(\mathbb{C}^*)^{n-m}$ with polytopes $\tilde{\pi}_2(\Delta_{m+1}), \dots, \tilde{\pi}_2(\Delta_k)$, and the variety X_i is mapped onto the variety of solutions of this system. $\square$

Proof of the reduction theorem. The reduction theorem reduces to Theorem 16. Indeed, for every $i \in J$, we can multiply the Laurent polynomial P_i by an appropriate monomial z^{q_i} in such a way that the Newton polytope Δ_i of the polynomial $z^{q_i} P_i$ lies in $M(\Delta_J)$. The variety X remains unchanged since the equations $P_i = 0$ and $z^{q_i} P_i = 0$ on the torus $(\mathbb{C}^*)^n$ are equivalent. Permuting the equations in the system if necessary, we can assume that $J = J_m$. By means of an automorphism of the torus $(\mathbb{C}^*)^n$, we can send all characters in $M(\Delta_J)$ to characters lying in the coordinate subspace $\mathbb{R}^m \subset \mathbb{R}^n$ on which all the coordinates with numbers $(m+1), \dots, n$ vanish. This reduces the reduction theorem to Theorem 16. $\square$

5.2. Theorems on the number of components. The following theorem is important for us.

Theorem 17 (the irreducibility theorem). *If the defect of every non-empty set J for the tuple of polytopes $\Delta_1, \dots, \Delta_k$ is positive, then the variety $X \subset (\mathbb{C}^*)^n$ defined by a sufficiently general system (1) is irreducible.*

In the rest of the paper we give a proof of this theorem and various sharpened versions of it. Now let us show that combining it with the reduction theorem enables us to compute the number of irreducible components of any variety determined by a system (1).

Definition. A subset $J \subset \{1, \dots, k\}$ is said to be *characteristic* for an independent tuple of bodies $\Delta_1, \dots, \Delta_k$ if the defect of J for this tuple is equal to zero and the defect of any larger set $J' \supset J$ is positive.

Lemma 18. *For every independent tuple of bodies there exists a characteristic set J.*

Proof. By definition, the defect of every set for an independent tuple of bodies is non-negative. There are sets with defect zero (for example, the empty set). Take the largest (with respect to inclusion) set with defect zero. A larger set can have neither negative nor zero defect. Hence its defect is positive. $\square$

Theorem 19. *If J is characteristic for $\Delta_1, \dots, \Delta_k$ under the hypotheses of the reduction theorem, then the varieties X_i are irreducible components of X (and their number is equal to $m!\, V(\Delta_{i_1}, \dots, \Delta_{i_m})$).*

Proof. We claim that the hypotheses of Theorem 17 hold for the tuple of polytopes $\{\pi_{J^\perp}(\Delta_i)\}$, $i \in J^\perp$. Indeed, by hypothesis, the defect of J is equal to zero, that is, $\dim \Delta_J = |J|$. For every non-empty subset $J^* \subset \{1, \dots, k\} \setminus J$ we put $\tilde{J} = J \cup J^*$. Since $J \subset \tilde{J}$ and J is characteristic, the defect of $\tilde{J}$ for the tuple $\Delta_1, \dots, \Delta_k$ is positive, that is, $\dim \Delta_{\tilde{J}} - |\tilde{J}| > 0$ or $(\dim \Delta_{\tilde{J}} - \dim \Delta_J) - (|\tilde{J}| - |J|) > 0$. It remains to note that the polytope $\sum_{i \in J^*} \pi_{J^\perp}(\Delta_i)$ has dimension $\dim \Delta_{\tilde{J}} - \dim \Delta_J$. Therefore J^* has positive defect for the tuple of polytopes $\{\pi_{J^\perp}(\Delta_i)\}$. By Theorem 17, the varieties X_i are irreducible. $\square$

§ 6. The numbers h^p of complete intersections

6.1. Holomorphic forms on compact complex varieties. Let M be a smooth compact n-dimensional complex algebraic variety. For every $p \geqslant 0$ we have a finite-dimensional space of holomorphic p-forms on M. Its dimension is denoted by $h^p(M)$. The *arithmetic genus* of M is defined as the number

$$\chi(M) = \sum_{p \geqslant 0} (-1)^p h^p(M).$$

(We use the definition in [6], formula (2). Note that a slightly different invariant of the variety is sometimes referred to as the arithmetic genus.) The number $h^0(M)$ is equal to the number of connected components of M. When $p > n$ we have $h^p(M) = 0$ since every holomorphic p-form on a complex n-dimensional manifold with $p > n$ is identically equal to zero.

Given a rational map $\pi\colon M_1 \to M_2$ between compact varieties M_1 and M_2, we see that the pullback $\pi^*\omega$ of a holomorphic form ω on M_2 is a holomorphic form on M_1. Hence the dimension $h^p(M)$ of the space of holomorphic p-forms and the arithmetic genus of a compact variety M are invariant under birational isomorphisms.

One can also define $h^p(M)$ for non-compact smooth algebraic varieties. This is done by declaring $h^p(M)$ to be equal to $h^p(\overline{M})$ for any smooth compactification $\overline{M}$ of M which is birationally equivalent to M. Such a compactification exists for every smooth M by Hironaka's theorem on the resolution of singularities.

For sufficiently general complete intersections X with fixed supports in a complex torus, the Newton polytopes yield an explicit construction of top-order forms holomorphic on some (and hence on any) smooth compactification $\overline{X}$ of X (see [1]). Exactly the same construction sometimes enables us to construct holomorphic forms on $\overline{X}$ of certain intermediate orders.

We recall the following definition (see [1], [2]). Let Δ be a convex integer polytope. Then $B^+(\Delta)$ *is the number of integer points lying strictly inside* Δ *in the topology of the minimal affine space* $L(\Delta)$ *containing* Δ.

The following theorem uses the notation of § 4.2: the varieties $\widetilde{X}_1$, X_1, X_2, X, the polytopes $\Delta_1, \dots, \Delta_k$, the sets J, $J^\perp$ and the number $l = |J|$ are as in § 4.2.

Theorem 20. *Suppose that*

1) $\dim \Delta_J = m$;

2) $B^+(\Delta_J) > 0$;

3) $\dim \Delta_{\tilde{J}} - |\tilde{J}| \geqslant \dim \Delta_J - |J|$ *for every set of indices* $\tilde{J} \subset \{1, \dots, k\}$ *that contains* J.

Then $h^{m-l}(X) > 0$.

Proof. Let $\widetilde{X}_1$ be the variety determined in $(\mathbb{C}^*)^m$ by the system (7). We define a form ω_{m-l} on $\widetilde{X}_1$ by the formula

$$\omega_{m-l} = z^q \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_m}{z_m} / dQ_{i_1} \wedge \dots \wedge dQ_{i_l}, \tag{8}$$

where q is any integer point lying strictly inside the polytope $\pi_{\Delta_J}(\Delta_J)$ (such a point exists since $B^+(\Delta_J) > 0$). By § 2 in [1], ω_{m-l} is holomorphic on some (and hence on any) smooth compactification of $\widetilde{X}_1$. The variety X_1 is equivalent to the product $\widetilde{X}_1 \times (\mathbb{C}^*)^{n-m}$, and this equivalence transforms the projection $\pi_{\Delta_j}\colon X_1 \to \widetilde{X}_1$ to the projection onto the first factor. The form $\pi^*_{\Delta_j}\omega_{m-l}$ is holomorphic on the compactification of X_1 equal to the product of any smooth compactifications of $\widetilde{X}_1$ and $(\mathbb{C}^*)^{n-m}$. Hence the form $\pi^*_{\Delta_j}\omega_{m-l}$ is holomorphic on any smooth compactification of X_1.

Let M be a sufficiently complete toric compactification of $(\mathbb{C}^*)^n$ for the tuple of polytopes $\Delta_1, \dots, \Delta_k$ (see [2] and § 7 below). By [2], the closures $\overline{X}_1$ and $\overline{X}_2$ of X_1 and X_2 in M are smooth varieties intersecting each other transversally, and their intersection $\overline{X}_1 \cap \overline{X}_2$ is a smooth closure of the variety $X = X_1 \cap X_2$. The form $\pi^*_{\Delta_j}\omega_{m-l}$ is holomorphic on $\overline{X}_1$ and hence on $\overline{X} \subset \overline{X}_1$. We see from (8) that ω_{m-l} never vanishes on $\widetilde{X}_1$. By Theorem 13, $\pi_{\Delta_j}(X)$ is dense in $\widetilde{X}_1$ and, therefore, the form $\pi^*_{\Delta_j}\omega_{m-l}$ is not identically equal to zero on X. Since this form is holomorphic on the closure $\overline{X}$ of X in M, it follows that $h^{m-l}(X) > 0$. $\square$

6.2. Cohomology and the numbers h^p. For every divisor D on a non-singular projective variety M we have a sheaf $\Omega(M, D)$ of germs of meromorphic functions f on M such that $(f) + D \geqslant 0$. If $D \leqslant 0$, then the germs of sections of this sheaf are germs of regular functions. We need only sheaves of this kind. The cohomology of M with coefficients in $\Omega(M, D)$ is denoted by $H^*(M, D)$. Omitting the zero divisor from the notation, we shall write $H^*(M)$ for the cohomology of M with coefficients in the sheaf of germs of regular functions. The cohomology $H^*(M)$ is responsible for the dimensions $h^p(M)$ of the spaces of holomorphic p-forms on M, namely, $h^p(M) = \dim H^p(M)$. In particular, $\dim H^0(M)$ is equal to the number of components of M and we have $H^p(M) = 0$ for $p > \dim M$.

We shall use the following notation. Vector subspaces $L_1, \dots, L_k$ are said to be *mutually transversal* in the ambient space L if the codimension in L of the intersection of any subtuple of these spaces is equal to the sum over this subtuple of their codimensions in L (in particular, if the sum of the codimensions of all subspaces is greater than the dimension of L, then the subspaces cannot be mutually transversal). Subvarieties $M_1, \dots, M_k$ are said to be *mutually transversal* in the ambient variety M if at every common point of any subtuple of these subvarieties their tangent spaces are mutually transversal in the tangent space to M at this point.

Let M be endowed with a fixed divisor M^∞ which is a union of smooth mutually transversal divisors M_j^∞. Consider the ring R of meromorphic functions on M whose restrictions to $M \setminus M_\infty$ are regular. We are interested in complete intersections in $M \setminus M^\infty$ given by systems of equations of the form $f_1 = \dots = f_k = 0$, where $f_i \in R$. Every divisor D on M can be expanded into a sum $D^0 + D^\infty$, where D^0 has no components with supports in M^∞ and the support of D^∞ lies in M^∞.

Let $(f) = D^0 + D^\infty$ be such an expansion of the divisor (f) of a function $f \in R$. The support of D^0 lies in the closure of the hypersurface which is defined in $M \setminus M^\infty$ by the equation $f = 0$. The divisor D^∞ is linearly equivalent to [4] $-D^0$. For our purposes it suffices to consider only those equations $f = 0$ for which the divisor D^∞

[4]Here and in what follows we use notation somewhat different from that in our references [1], [2]. The main difference is in the signs. The divisors and cohomology which we denote by D^∞, Δ^∞ and $H^*(M, \Delta^\infty)$, are denoted in [1], [2] by $-D_\infty$, $-\{\Delta\}$ and $H^*(M, \{-\Delta\})$ respectively.

in the expansion of (f) is negative. In what follows we always assume that this condition holds.

Consider a complete intersection in $M\setminus M^\infty$ given by a system $f_1=\dots=f_k=0$, $f_i\in R$, and let $(f_i)=D_i^0+D_i^\infty$ be the expansions of the divisors of the functions f_i. Suppose that $D_1^0,\dots,D_k^0$ are non-singular divisors in M and their intersection is non-empty. We also assume that all the divisors D_i^0 and M_j^∞ (where M_j^∞ are the components of M^∞) are mutually transversal. Substantial information on the numbers h^p of the complete intersection can be obtained if we know the dimensions of the cohomology groups $H^*(M,n_1D_1^\infty+\dots+n_kD_k^\infty)$, $n_i=\{0,1\}$, of the ambient manifold M. We recall how to do this [6].

6.3. Exact sequences. For every m, $1\leqslant m\leqslant k$, we put $M_m=D_1^0\cap\dots\cap D_m^0$. Each variety in the sequence $M=M_0\supset\dots\supset M_k$ is a hypersurface in the previous variety. For every divisor $D^\infty\leqslant 0$ and every m, $1\leqslant m\leqslant k$, we consider the exact sequence of sheaves

$$0\to\Omega(M_{m-1},D^\infty-D_m^0)\xrightarrow{i}\Omega(M_{m-1},D^\infty)\xrightarrow{j}\hat\Omega(M_{m-1},D^\infty)\to 0. \tag{9}$$

Here $\Omega(M_{m-1},D^\infty-D_m^0)$ is the sheaf on M_{m-1} associated with the divisor cut out on M_{m-1} by the divisor $D^\infty-D_m^0$ (it follows from our assumptions that the variety M_{m-1}, the support of D_m^0 and the components M_j^∞ of the support of D^∞ are mutually transversal). The sheaves $\Omega(M_{m-1},D^\infty)$ and $\Omega(M_m,D^\infty)$ are defined in a similar way. The sheaf $\hat\Omega(M_{m-1},D^\infty)$ is the trivial extension of $\Omega(M_m,D^\infty)$ to a sheaf on M_{m-1}. The homomorphism i is an embedding and the homomorphism j is trivial at every point $a\in M_{m-1}\setminus M_m$ and sends each germ of a function on M_{m-1} to its restriction to M_m at every point $a\in M_m$. The corresponding cohomology exact sequence can be written in the form

$$0\to H^0(M_{m-1},D^\infty+D_m^\infty)\to H^0(M_{m-1},D^\infty)\to H^0(M_m,D^\infty)\to\cdots \tag{10}$$

(since the cohomology groups of the sheaves $\Omega(M_{m-1},D^\infty-D_m^0)$ and $\Omega(M_{m-1},D^\infty+D_m^\infty)$, as well as of $\hat\Omega(M_m,D^\infty)$ and $\Omega(M_{m-1},D^\infty)$, are canonically isomorphic). Given any variety $\mathcal{M}$ and any divisor $\mathcal{D}$ on $\mathcal{M}$, we write $\chi(\mathcal{M},\mathcal{D})$ for the Euler characteristic of $\mathcal{M}$ with coefficients in $\Omega(\mathcal{M},\mathcal{D})$. The Euler characteristic of a variety with coefficients in a sheaf is equal to the sum of its Euler characteristics with coefficients in a subsheaf and the quotient sheaf. The exact sequences (10) enable us to find the numbers $\chi(M_k,D^\infty)$. Here is the answer in the case when $D^\infty=0$ (we are interested only in the number $\chi(M_k)$).

Theorem 21. *The arithmetic genus $\chi(M_k)$ of the variety M_k is equal to*

$$\chi(M)-\sum_i\chi(M,D_i^\infty)+\sum_{i<j}\chi(M,D_i^\infty+D_j^\infty)-\dots+(-1)^k\chi\biggl(M,\sum_{1\leqslant i\leqslant k}D_i^\infty\biggr).$$

For every non-empty set $J\subset\{1,\dots,k\}$ we put $D_J^\infty=\sum_{i\in J}D_i^\infty$.

Theorem 22. *The following bound holds*:

$$h^i(M_k)\leqslant h^i(M)+\sum_{J\neq\varnothing}\dim H^{i+|J|}(M,D_J^\infty).$$

Proof. We rewrite this bound, generalize it, and prove the generalized bound. When $J = \varnothing$ we put $D^\infty_J = 0$. Then the bound in the theorem may be rewritten in the form $h^i(M_k) \leqslant \sum_J \dim H^{i+|J|}(M, D^\infty_J)$ (the sum is taken over all subsets J, including $J = \varnothing$) since $\dim H^i(M, 0) = h^i(M)$. The following more general inequality coincides with the desired one when $D^\infty = 0$: for every divisor $D^\infty \leqslant 0$ we have

$$\dim H^i(M_k, D^\infty) \leqslant \sum_J \dim H^{i+|J|}(M, D^\infty + D^\infty_J).$$

Let us prove this inequality by induction on k. For every divisor $D^\infty \leqslant 0$ and any integers $j \geqslant 0$ and m, $1 \leqslant m \leqslant k$, the part

$$\to H^j(M_{m-1}, D^\infty) \to H^j(M_m, D^\infty) \to H^{j+1}(M_{m-1}, D^\infty + D^\infty_m) \to \cdots$$

of the exact sequence (10) yields that

$$\dim H^j(M_m, D^\infty) \leqslant \dim H^j(M_{m-1}, D^\infty) + \dim H^{j+1}(M_{m-1}, D^\infty + D^\infty_m). \quad (11)$$

When $k = 1$, the desired assertion coincides with (11) with $j = i$ and $m = 1$. Suppose that the theorem is true for $k - 1$.

For every $J \subset \{1, \ldots, k\}$ put $J^* = J \cap \{1, \ldots, k-1\}$. Then either $J = J^*$, or $J = J^* \cup \{k\}$. In the first case,

$$|J| = |J^*|, \qquad D^\infty + D^\infty_J = D^\infty + D^\infty_{J^*}. \quad (12)$$

In the second case,

$$|J| = |J^*| + 1, \qquad D^\infty + D^\infty_J = D^\infty + D^\infty_{J^*} + D^\infty_k. \quad (13)$$

By the inductive hypothesis for the non-negative divisor D^∞ and every $i \geqslant 0$ we have

$$\dim H^i(M_{k-1}, D^\infty) \leqslant \sum H^{i+|J^*|}(M, D^\infty + D^\infty_{J^*}), \quad (14)$$

where the sum is taken over all subsets $J^* \subset \{1, \ldots, k-1\}$.

By the inductive hypothesis for every divisor $D^\infty + D^\infty_k \leqslant 0$ and the number $i + 1$ we have

$$\dim H^{i+1}(M_{k-1}, D^\infty + D^\infty_k) \leqslant \sum \dim H^{|J^*|+i+1}(M, D^\infty_{J^*} + D^\infty_k + D^\infty), \quad (15)$$

where the sum is taken over all subsets $J^* \subset \{1, \ldots, k-1\}$.

Substituting the right-hand sides of (14) and (15) for $\dim H^i(M_{k-1}, D^\infty)$ and $\dim H^{i+1}(M_{k-1}, D^\infty + D^\infty_k)$ in (11) and using (12) and (13), we obtain the desired inequality for the parameters k, i and the divisor $D^\infty \leqslant 0$. □

Corollary 23. *Let i be such that for every $J \neq \varnothing$ we have $H^{i+|J|}(M, D^\infty_J) = 0$. Then $h^i(M_k) \leqslant h^i(M)$.*

§ 7. Toric compactifications

We recall some results on toric compactifications (see [7]) and their applications to the theory of Newton polytopes (see [1], [2]). The torus $(\mathbb{C}^*)^n$ induces not only the character space $\mathbb{R}^n$, but also the *space of one-parameter semigroups* $(\mathbb{R}^n)^*$ endowed with a lattice $(\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^*$. Each integer point $m = (m_1, \dots, m_n) \in (\mathbb{Z}^n)^*$ determines a one-parameter group, namely, the homomorphism $t^m\colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ sending every point $t \in \mathbb{C}^*$ to the point $(t^{m_1}, \dots, t^{m_n}) \in (\mathbb{C}^*)^n$. The composite of the homomorphism $t^m\colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ and a character $z^k\colon (\mathbb{C}^*)^n \to \mathbb{C}^*$, where $z^k = z_1^{k_1} \cdots z_n^{k_n}$, is a homomorphism $t^{\langle k,m\rangle}\colon \mathbb{C}^* \to \mathbb{C}^*$, where $\langle k, m\rangle = k_1m_1 + \dots + k_nm_n$. The pairing $\langle k, m\rangle$ extends by continuity to the character space and the space of one-parameter groups and establishes a duality between them.

A *finite rational cone* in $(\mathbb{R}^n)^*$ is a subset of $(\mathbb{R}^n)^*$ determined by finitely many linear inequalities $\{\langle x, m_j\rangle \leqslant 0\}$, where the $m_j \in \mathbb{Z}^n$ are integer points of the dual space $\mathbb{R}^n$. Each face of a finite rational cone is also a finite rational cone.

A *complete fan* in $(\mathbb{R}^n)^*$ is a finite set $F = \{\sigma_i\}$ of finite rational cones σ_i with the following properties.

1) $\bigcup \sigma_i = (\mathbb{R}^n)^*$.

2) The cones $\sigma_i \in F$ contain no vector subspaces of positive dimension.

3) For every pair of cones $\sigma_i, \sigma_j \in F$ their intersection $\sigma_i \cap \sigma_j$ lies in F and is a face of σ_i and a face of σ_j.

A *toric compactification* is a normal complete algebraic variety M containing the torus $(\mathbb{C}^*)^n$ such that the group action of $(\mathbb{C}^*)^n$ on itself extends algebraically to M. The action of $(\mathbb{C}^*)^n$ decomposes every toric compactification M into finitely many orbits $\{O_i\}$, exactly one of which is n-dimensional (the torus $(\mathbb{C}^*)^n$).

For every toric compactification M one can define a complete fan F_M in the space of one-parameter groups and establish a one-to-one correspondence $\rho\colon F_M \to \{O_i\}$ between F_M and the set of orbits $\{O_i\}$ in M in such a way that the following conditions hold.

1) $\dim_{\mathbb{R}} \sigma = n - \dim_{\mathbb{C}} \rho(\sigma)$.

2) If $m \in \sigma$ is an integer point lying inside σ (in the topology of the minimal affine space containing σ), then $\lim_{t\to 0} t^m \in \rho(\sigma)$.

The map $M \to F_M$ establishes a one-to-one correspondence between toric compactifications and complete fans.

One-dimensional cones in F_M correspond under the map ρ to $(n-1)$-dimensional orbits in M. Each one-dimensional cone is generated by a primitive integer vector m. Hence the *primitive integer vector $m(O)$ corresponding to an $(n-1)$-dimensional orbit O* is well defined in the space of one-parameter groups. The order of the character z^k on an $(n-1)$-dimensional orbit O is equal to $\langle k, m(O)\rangle$.

The support $\operatorname{supp}(P)$ of a Laurent polynomial P lies in the character lattice and its Newton polytope $\Delta(P)$ lies in the character space. On the dual space of one-parameter groups we have a function H_P sending each covector m to the number $\min_{k\in\operatorname{supp}(P)}\langle m, k\rangle$. The function H_P coincides with the *support function* of the polytope $\Delta = \Delta(P)$, which is defined by the formula $H_\Delta(m) = \min_{k\in\Delta}\langle m, k\rangle$. The order $\operatorname{ord} P|_O$ of the Laurent polynomial P on an $(n-1)$-orbit O is equal to $H_P(m(O))$ and depends only on the polytope $\Delta = \Delta(P)$. We write $\overline{O}$ for the divisor which is the closure of an orbit O of dimension $(n-1)$.

With every Laurent polynomial P we associate a divisor P^∞ on M by the formula $P^\infty = \sum \operatorname{ord} P|_O \overline{O}$, where the sum is taken over all $(n-1)$-dimensional orbits O. The divisor P^∞ is invariant under the toric action on M and depends only on the polytope $\Delta = \Delta(P)$. In what follows we denote the divisor by Δ^∞.

Remark. The equations $P = 0$ and $z^q P = 0$ on the torus $(\mathbb{C}^*)^n$ are equivalent. If $-q \in \Delta(P)$, then $0 \in \Delta(z^q P)$. Hence for our purposes it suffices to consider only those polytopes Δ which contain the point 0. For such polytopes, the support function $\Delta(z^q P)$ is less than or equal to zero and we have $\Delta^\infty \leqslant 0$ (see footnote 4).

Definition. A complete fan F is said to be *sufficiently complete* for a polytope Δ if the support function H_Δ is linear on each cone $\sigma_i \in F$.

A polytope Δ induces a decomposition of the dual space into equivalence classes, where two covectors are *equivalent* if the restrictions of the linear functions determined by them attain their minimum values at the same face of Δ. With each integer polytope Δ we associate the fan $\Delta^\perp$ whose cones are the closures of these equivalence classes. A fan F is sufficiently complete for Δ if and only if it is a subdivision of $\Delta^\perp$.

Since the divisor Δ^∞ is invariant under the torus action, the theory of toric varieties enables us to compute the dimensions of the cohomology groups $H^*(M, \Delta^\infty)$ (see, for example, [7]). The answer is especially simple in the case when the fan F_M is sufficiently complete for the polytope Δ.

We recall that $B^+(\Delta)$ *is the number of integer points lying strictly inside* Δ *in the topology of the minimal affine space* $L(\Delta)$ *containing* Δ. The following theorem holds (see the theorem in § 4 of [1] and footnote 4).

Theorem 24. *Suppose that F_M is sufficiently complete for Δ. Then $H^i(M, \Delta^\infty) = 0$ when $i \neq d$, where $d = \dim \Delta$, and $\dim H^d(M, \Delta^\infty) = B^+(\Delta)$.*

If the fan F_M is sufficiently complete for $T = \Delta_1 + \dots + \Delta_k$, then it is sufficiently complete for every polytope $n_1\Delta_1 + \dots + n_k\Delta_k$ with $n_i \geqslant 0$.

Corollary 25. *If F_M is sufficiently complete for T, then for every polytope $\Delta = n_1\Delta_1 + \dots + n_k\Delta_k$ with $n_i \geqslant 0$ the dimensions of the cohomology groups $H^*(M, \Delta^\infty)$ are those calculated in Theorem 24.*

Corollary 26. *Suppose that F_M is sufficiently complete for Δ. Then the Euler characteristic of M with coefficients in $\Omega(M, \Delta^\infty)$ is equal to $B(\Delta)$, where by definition $B(\Delta) = (-1)^{\dim \Delta} B^+(\Delta)$.*

Corollary 27. *For every smooth projective toric variety M we have $h^0(M) = 1$ and the numbers $h^i(M)$, $i > 0$, are equal to zero.*

Proof. The numbers h^i are birational invariants. All n-dimensional toric varieties are birationally equivalent to $\mathbb{C}P^n$, for which the corollary obviously holds. Let us also deduce the corollary directly from Theorem 24. Let $\Delta = \{0\}$ be the polytope consisting of the point 0. Then $\dim \Delta = 0$ and $B^+(\Delta) = 1$. The complete fan F_M

of any toric variety M is sufficiently complete for $\Delta = \{0\}$, and the divisor Δ^∞ on M is equal to zero in this case. It remains to use Theorem 24 with $\Delta = \{0\}$. $\square$

It is known (see, for example, [7]) that a toric compactification M is a smooth projective variety if and only if its fan F_M possesses the following properties.

1) Every cone $\sigma \in F_M$ is a simplicial cone.

2) For every $\sigma \in F_M$ the primitive integer vectors lying on the edges of σ generate the lattice $\mathbb{Z}^n \cap L(\sigma)$, where $L(\sigma)$ is the minimal vector space containing σ.

3) The fan F_M is dual to some integer polytope Δ (that is, $F_M = \Delta^\perp$).

It is known that every complete fan F can be subdivided (in infinitely many ways) to obtain the fan of a smooth projective variety (see, for example, [7]).

§ 8. Proof of the irreducibility theorem

Here we use the results of §§ 6 and 7 to prove the irreducibility theorem. We also make some calculations related to the numbers $h^p(X)$ of an algebraic variety X determined by the system (1).

Let $\Delta_1, \dots, \Delta_k$ be the Newton polytopes of the Laurent polynomials $P_1, \dots, P_k$ occurring in the system (1). We shall assume that this tuple of polytopes is independent (otherwise a generic system of equations (1) is incompatible). In this section M stands for any (fixed) smooth projective toric compactification of $(\mathbb{C}^*)^n$ such that F_M is a subdivision of $\Delta^\perp$, where $\Delta = \Delta_1 + \dots + \Delta_k$. The union of the closures of the $(n-1)$-dimensional orbits in M is denoted by M^∞. Since M is a smooth toric variety, M^∞ is a divisor with normal crossings (see, for example, [7]). The open variety $M \setminus M^\infty$ is the torus $(\mathbb{C}^*)^n$, and the ring R of meromorphic functions on M whose restrictions to the torus are regular coincides with the ring of Laurent polynomials. We preserve the notation of § 6, having in mind the varieties M, M^∞ and R introduced above. To use the results of §§ 6 and 7 for the variety X determined by a sufficiently general system (1), we need the following lemma.

Lemma 28. *Let P_i be any of the Laurent polynomials occurring in a sufficiently general system* (1). *Then the closure D_i^0 of the hypersurface $P_i = 0$ in M is a smooth hypersurface in M. Moreover, the divisors D_i^0 and the closures of all $(n-1)$-dimensional orbits are mutually transversal in M.*

Proof. For every non-empty subset $J = \{i_1, \dots, i_l\}$ of the set $\{1, \dots, k\}$ we consider the corresponding subsystem (5) of the system (1). The following conditions on the tuple of polynomials $P_1, \dots, P_k$ are sufficient (and necessary) for the conclusion of Lemma 28 to hold. For every J the subsystem (5) must be Δ-*non-degenerate* (see [2]) for its tuple of Newton polytopes $\Delta_{i_1}, \dots, \Delta_{i_l}$. This assertion follows automatically from the theorem in § 2 of [2]. $\square$

Suppose that the system (1) is compatible and sufficiently general for Lemma 28 to hold. Then the closure in M of the variety X of solutions of (1) is an intersection of smooth divisors $D_1^0, \dots, D_k^0$ such that the corresponding divisors $D_1^\infty, \dots, D_k^\infty$ are equal to $\Delta_1^\infty, \dots, \Delta_k^\infty$. Hence the results of § 6 are applicable to this closure. To apply them, we need some information on the dimensions of the cohomology groups $H^*(M, n_1\Delta_1^\infty + \dots + n_k\Delta_k^\infty)$, $n_i = \{0, 1\}$. This information is contained in Corollary 25 of Theorem 24.

Theorem 29. *If X is given by the system* (1), *then the arithmetic genus $\chi(X)$ satisfies*

$$\chi(X)=1-\sum_i B(\Delta_i)+\sum_{i<j} B(\Delta_i+\Delta_j)-\sum_{i<j<l} B(\Delta_i+\Delta_j+\Delta_l)+\cdots. \qquad (16)$$

Proof. This follows automatically from Theorem 21 and Corollary 26. □

When $k=n$, the right-hand side of (16) is equal to $n!\,V(\Delta_1,\dots,\Delta_n)$ (see [1]). This proves the Bernstein–Kushnirenko theorem (if $\dim X=0$, then $\chi(X)$ is the number of points in X).

Remark. The formula for $\chi(X)$ and the deduction of the Bernstein–Kushnirenko theorem from it were given in [1].

Theorem 30. *For every non-negative integer i we have*

$$h^i(X)\leqslant \sum_{\{J\mid \dim\Delta_J-|J|=i\}} B^+(\Delta_J)+\delta_0^i,$$

where $\delta_0^i=0$ for $i\neq 0$ and $\delta_0^0=1$, and the sum is taken over all $J\neq\varnothing$.

Proof. This follows automatically from Theorems 22, 24 and the equality $h^i(M)=\delta_0^i$ (see Corollary 27). □

Definition. We say that an integer $i\geqslant 0$ is *critical* for a tuple $\Delta_1,\dots,\Delta_k$ of independent integer polytopes in $\mathbb{R}^n$ if there is a non-empty subset $J\subset\{1,\dots,k\}$ such that $B^+(\Delta_J)>0$ and $\dim\Delta_J-|J|=i$.

Suppose that X is given by a sufficiently general system (1) with independent Newton polytopes $\Delta_1,\dots,\Delta_k$.

Theorem 31. *Let i be non-critical for $\Delta_1,\dots,\Delta_k$. If $i=0$, then X is irreducible. But if $i>0$, then $h^i(X)=0$.*

Proof. For non-critical i, the inequality in Theorem 30 takes the form $h^i(X)\leqslant\delta_0^i$. But the number $h^0(X)$, which coincides with the number of components of X, is strictly positive and the numbers $h^i(X)$ are non-negative. □

Theorem 17 (the irreducibility theorem) follows from Theorem 31. Theorem 30 yields the following strengthening of a result in [1].

Corollary 32. *If all numbers i with $0\leqslant i<n-k$ are non-critical for $\Delta_1,\dots,\Delta_k$, then $h^0(X)=1$, $h^{(n-k)}(X)=(-1)^{(n-k)}(\chi(X)-1)$ and $h^p(X)=0$ for $p\neq 0$, $p\neq n-k$.*

Proof. By Theorem 30 we have $h^0(X)=1$ and $h^p=0$ for $0<p<n-k$. Since the dimension of X is equal to $n-k$, we also have $h^p=0$ for $n-k<p$. □

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