ON SOME PROBLEMS SUGGESTED BY THE TRACE FORMULA

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In the present theory of automorphic representations, a major goal is to stabilize the trace formula. Its realization will have important consequences, among which will be the proof of functoriality in a significant number of cases. However, it will require much effort, for there are a number of difficult problems to be solved first. Some of the problems, especially those concerning orbital integrals, were studied in [9(e)]. They arise when one tries to interpret one side of the trace formula. The other side of the trace formula leads to a different set of problems. Among these, for example, are questions relating to the nontempered automorphic representations which occur discretely. Our purpose here is to describe some of these problems and to suggest possible solutions.

Some of the problems have in fact been formulated as conjectures. They have perhaps been stated in greater detail than is justified, for I have not had sufficient time to ponder them. However, they seem quite natural to me, and I will be surprised if they turn out to be badly off the mark.

Our discussion will be rather informal. We have tried to keep things as simple as possible, sometimes at the expense of omitting pertinent details. Section 1, which is devoted to real groups, contains a review of known theory, and a description of some problems and related examples. Section 2 has a similar format, but is in the global setting. We would have liked to follow it with a detailed discussion of the trace formula, as it pertains to the conjecture in Section 2. However, for want of time, we will be much briefer. After opening with a few general remarks, we will attempt in Section 3 to motivate the conjecture with the trace formula only in the case of $PSp(4)$. In so doing, we will meet a combinatorial problem which is trivial for $PSp(4)$, but is more interesting for general groups.

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§1. A PROBLEM FOR REAL GROUPS

1.1. The trace formula, which we will discuss presently, is an equality of invariant distributions. The study of such distributions leads to questions in local harmonic analysis. We will begin by looking at one such question over the real numbers.

For the time being, we will take G to be a reductive algebraic group defined over $\mathbb{R}$. For simplicity we shall assume that G is quasi-split. Let $\prod(G(\mathbb{R}))$ (resp. $\prod_{temp}(G(\mathbb{R}))$) denote the set of equivalence classes of irreducible representations (resp. irreducible tempered representations) of $G(\mathbb{R})$. In the data which one feeds into the trace formula are functions f in $C_c^\infty(G(\mathbb{R}))$. Since the terms of the trace formula are invariant distributions, we need only specify f by its values on all such distributions.

<u>Theorem 1.1.1</u>: The space of invariant distributions on $G(\mathbb{R})$ is the closed linear span of

$$\{tr(\pi): \prod_{temp}(G(\mathbb{R})),$$

where $tr(\pi)$ stands for the distribution $f \to tr\pi(f)$.

One can establish this theorem from the characterization [1(a)] of the image of the Schwartz space of $G(\mathbb{R})$ under the (operator valued) Fourier transform. We hope to publish the details elsewhere.

Thus, for the trace formula, we need only specify the function

$$(1.1.2) \qquad F(\pi) = \operatorname{tr} \pi(f), \qquad \pi \in \Pi_{temp}(G(\mathbb{R})).$$

It is clearly important to know what functions on $\Pi_{temp}(G(\mathbb{R}))$ are of this form. The elements in $\Pi_{temp}(G(\mathbb{R}))$ can be given by a finite number of parameters, some continuous and some discrete. Via these parameters, one can define a Paley-Wiener space on $\Pi_{temp}(G(\mathbb{R}))$. It consists of functions which, among other things, are in the classical Paley-Wiener space in each continuous parameter. We would expect this Paley-Wiener space on $\Pi_{temp}(G(\mathbb{R}))$ to be the image of $C_c^\infty(G(\mathbb{R}))$ under the map above. This fact may well be a consequence of recent work of Clozel and Delorme. We shall assume it implicitly in what follows.

There is one point we should mention before going on. The function F can be evaluated on any invariant distribution on $G(\mathbb{R})$. In particular,

$$F(\pi) = \langle \operatorname{tr} \pi, F\rangle = \operatorname{tr} \pi(f)$$

is defined for any irreducible representation π, and not just a tempered one. If $\rho = \oplus \pi_i$ is a finite sum of irreducible representations, we set

$$F(\rho) = \sum F(\pi_i).$$

Now, consider an induced representation

$$\rho_\sigma = \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} (\sigma \otimes id_N),$$

where $P = NM$ is a parabolic subgroup of G (defined over $\mathbb{R}$), σ is a representation in $\Pi_{temp}(M(\mathbb{R}))$, and id_N is the trivial representation of the unipotent radical $N(\mathbb{R})$. Let λ be a complex valued linear function on $\mathfrak{a}_M$, the Lie algebra of the split component of the center of $M(\mathbb{R})$, and let σ_λ be the representation obtained by translating σ by λ. Then ρ_{σ_λ} is in general a nonunitary, reducible representation of $G(\mathbb{R})$. Representations of this form are sometimes called standard representations. The function $F(\rho_{\sigma_\lambda})$, defined by the prescription above, can be obtained by analytic continuation from the purely imaginary values of λ, where the induced representation is tempered. Suppose that π is an arbitrary irreducible, but not necessarily tempered, representation of $G(\mathbb{R})$. It is known (see [15] that $tr(\pi)$ can be written

$$(1.1.3) \qquad tr(\pi) = \sum_\rho M(\pi,\rho)tr(\rho),$$

where ρ ranges over a finite set of standard representations of $G(\mathbb{R})$ and $\{M(\pi,\rho)\}$ is a uniquely determined set of integers. Then $F(\pi)$ is given by

$$F(\pi) = \sum_\rho M(\pi,\rho)F(\rho).$$

Thus, the problem of determining $F(\pi)$ is equivalent to determining the decomposition (1.1.3).

1.2. Among the invariant distributions are the stable distributions, which are of particular interest for global applications. Shelstad has shown [11(c)] that these may be defined either by orbital integrals or, as we shall do, by tempered characters.

We recall the Langlands classification [9(a)] of $\Pi(G(\mathbb{R}))$. Let $\Phi(G/\mathbb{R})$ be the set of admissible maps

$$\phi: W_{\mathbb{R}} \to {}^LG,$$

where $W_{\mathbb{R}}$ is the Weil group of $\mathbb{R}$, and

$${}^LG = {}^LG^0 \times W_{\mathbb{R}}$$

is the L-group of G. The elements in $\Phi(G/\mathbb{R})$ are to be given only up to conjugacy by ${}^LG^0$. To each $\phi \in \Phi(G/\mathbb{R})$ Langlands associates an L-packet $\Pi_\phi = \Pi_\phi^G$ consisting of finitely many representations in $\Pi(G(\mathbb{R}))$. He shows that the representations in Π_ϕ are tempered if and only if the projection of the image of ϕ onto ${}^LG^0$ is bounded. Let $\Phi_{temp}(G/\mathbb{R})$ denote the set of all such ϕ.

Definition 1.2.1: A stable distribution is any distribution, necessarily invariant, which lies in the closed linear span of

$$\{\sum_{\pi\in\Pi_\phi}\mathrm{tr}(\pi): \phi \in \Phi_{temp}(G/\mathbb{R})\}.$$

If F is a function of the form (1.1.2), we can set

$$F(\phi) = \sum_{\pi\in\Pi_\phi} F(\pi)$$

for any $\phi \in \Phi_{temp}(G/\mathbb{R})$. In [11(c)] Shelstad shows that any tempered character on $G(\mathbb{R})$ can be expressed in terms of sums of this form, but associated to some other groups of lower dimension. Given our discussion above, this means that any invariant distribution on $G(\mathbb{R})$ may be expressed in terms of stable distributions associated to other groups. We shall review some of this theory.

The notion of endoscopic group was introduced in [9(c)] and studied further in [11(c)]. Let s be a semisimple element in ${}^LG^0$, defined modulo

$$Z_G = \mathrm{Cent}({}^LG, {}^LG^0),$$

the centralizer of ${}^LG$ in ${}^LG^0$. An endoscopic group $H = H_s$ for G (over $\mathbb{R}$) is a quasi-split group in which ${}^LH^0 = {}^LH_s^0$ equals

$$\mathrm{Cent}(s, {}^LG^0)^0,$$

the connected component of the centralizer of s in ${}^LG^0$. If G is a split group with trivial center, this specifies H uniquely. For then ${}^LG^0$ is a simply connected complex group, in which the centralizer of any semisimple element is connected ([14], Theorem 2.15). The group H is then the unique split group whose L group is the direct product of ${}^L H^0$ with $W_{\mathbb{R}}$. In general, it is required only that each element $w \in W_{\mathbb{R}}$ act on ${}^L H^0$ by conjugation with some element

$$g \times w, \qquad g \in {}^L G^0,$$

in $\mathrm{Cent}(s, {}^L G)$. Since the group $\mathrm{Cent}(s, {}^L G^0)$ is not in general connected, there might be more than one endoscopic group for a given s and ${}^L H_s^0$. Two endoscopic groups H_s and $H_{s'}$ will be said to be equivalent if there is a $g \in {}^L G^0$ such that s equals $gs'g^{-1}$ modulo the product of Z_G with the connected component of Z_{H_s}, and the map

$$\mathrm{ad}(g^{-1}) : {}^L H^0 \to {}^L (H')^0$$

commutes with the action of $W_{\mathbb{R}}$. (Thus, for us an endoscopic group really consists of the element s as well as the group H, and should strictly be called an endoscopic datum. See [9(e)].)

An admissible embedding ${}^L H \subset {}^L G$ of an endoscopic group is one which extends the given embedding of ${}^L H^0$, which commutes with the projections onto $W_{\mathbb{R}}$, and for which the image of ${}^L H$ lies in $\mathrm{Cent}(s, {}^L G)$. We shall suppose from now on that for each endoscopic group we have fixed an admissible embedding ${}^L H \subset {}^L G$, such that the embeddings for equivalent groups are compatible. (The additional restriction this puts on G is not serious. See [9(c)].) We shall say that H is <u>cuspidal</u> if the image of ${}^L H$ in ${}^L G$ lies in no proper parabolic subgroup of ${}^L G$.

<u>Example 1.2.2</u>: Let $G = PSp(4)$. Then

$${}^L G^0 = Sp(4,\mathbb{C}) = \{ g \in GL(4,\mathbb{C}) : \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} = g \}.$$

The only cuspidal endoscopic groups are G and H_s, with

$$s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$ Then

$$ {}^L H_s^0 = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \right\} \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) , $$

and

$$ H_s \cong PGL(2) \times PGL(2) . $$

For each of these groups we take the obvious embedding of ${}^L H$ into ${}^L G$.

If ϕ is any parameter in $\Phi(G/\mathbb{R})$, define

$$ C_\phi = C_\phi^G = \mathrm{Cent}(\phi(W_{\mathbb{R}}), {}^L G^0) , $$

the centralizer in ${}^L G^0$ of the image of ϕ. Since the homomorphism ϕ is determined only up to ${}^L G^0$ conjugacy, C_ϕ is really only a conjugacy class of subgroups of ${}^L G^0$. However, we can identify each of these subgroups with a fixed abstract group, the identification being canonical up to an inner automorphism of the given group. Set

$$ \mathcal{C}_\phi = C_\phi / C_\phi^0 Z_G , $$

where C_ϕ^0 is the identity component of C_ϕ. Then $\mathcal{C}_\phi$ is a finite group which is known to be abelian. ([11(c)]. See also [5].) It can therefore be canonically identified with an abstract group which depends only on the class of ϕ.

For each $\phi \in \Phi_{temp}(G(\mathbb{R}))$, Shelstad defines a pairing $<, >$ on $\prod_\phi \times \mathcal{C}_\phi$, such that the map

$$ \pi \to <\pi, \cdot>, \qquad \pi \in \prod\nolimits_\phi , $$

is an injection from $\prod_\phi$ into the group $\hat{\mathcal{C}}_\phi$ of characters of $\mathcal{C}_\phi$. Unfortunately, the pairing cannot be defined canonically. However.

Shelstad shows that there is a function c from C_ϕ/Z_G to $\{\pm 1\}$, which is invariant on conjugacy classes, such that

$$c(s)\langle\bar{s},\pi\rangle, \qquad\qquad s\in C_\phi/Z_G,\ \pi\in\Pi_\phi,$$

is independent of the pairing. Here, $\bar{s}$ is the projection of s onto C_ϕ. This latter function can be used to map functions on $G(\mathbb{R})$ to functions on endoscopic groups.

Given a parameter $\phi\in\Phi_{temp}(G/\mathbb{R})$ and a semisimple element $s\in C_\phi/Z_G$, one can check that there is a unique endoscopic group $H=H_s$ such that

$$\phi(W_{\mathbb{R}})\subset {}^LH\subset {}^LG\ .$$

ϕ then defines a parameter $\phi_1\in\Phi_{temp}(H/\mathbb{R})$. For a given H, every parameter in $\Phi_{temp}(H/\mathbb{R})$ arises in this way. For any function $f\in C_c^\infty(G(\mathbb{R}))$, Shelstad defines a function $f_H\in C_c^\infty(H(\mathbb{R}))$, unique up to stable distributions on $H(\mathbb{R})$. To do so, it is enough to specify the value

$$f_H(\phi) \;=\; \sum_{\pi\in\Pi^H_{\phi_1}} f_H(\pi_1) \;=\; \sum_{\pi_1\in\Pi^H_{\phi_1}} \mathrm{tr}\ \pi_1(f_H)\ ,$$

for every such ϕ_1. This is done by setting

$$f_H(\phi_1) \;=\; c(s)\sum_{\pi\in\Pi_\phi}\langle\bar{s},\pi\rangle\ \mathrm{tr}\ \pi(f). \tag{1.2.3}$$

Actually, Shelstad defines f_H by transferring orbital integrals, and then proves the formula (1.2.3) as a theorem. However, we shall take the formula as a definition. Shelstad shows that the mapping $f\to f_H$ is canonically defined up to a sign. (It also depends on the embedding ${}^LH\subset {}^LG$ which we have fixed.) We shall fix the signs in any way, asking only that in the case $H=G$, f_G be consistent with the notation above. That is, $c(1) = 1$.

1.3. It is important for the trace formula to understand how the notions above relate to <u>nontempered</u> parameters ϕ. Shelstad defined the pairings $\langle \bar{s}, \pi \rangle$ only for tempered ϕ, but it is easy enough to extend the definition to arbitrary parameters. For one can show that there is a natural way to decompose any parameter ϕ by

$$\phi(w) = \phi_0(w)\phi_+(w), \qquad \phi_0 \in \Phi_{\mathrm{temp}}(G/\mathbb{R}), \ \phi_+ \in \Phi(G/\mathbb{R})),$$

so that the images of ϕ_0 and ϕ_+ commute, and so that ϕ itself is tempered whenever $\phi_+(W_{\mathbb{R}}) = \{1\}$. The centralizer in ${}^LG$ of the image of ϕ_+ will be the Levi component ${}^LM$ of a parabolic subgroup of ${}^LG$, and $\prod^M_{\phi_+}$ will consist of a positive quasi-character ν_+ of $M(\mathbb{R})$. The image of ϕ_0 must lie in ${}^LM$, so that ϕ_0 defines an element in $\Phi_{\mathrm{temp}}(M/\mathbb{R})$. There will be a bijection between $\prod^M_{\phi_0}$ and $\prod^G_{\phi}$, the elements in $\prod^G_{\phi}$ being the Langlands quotients obtained from the tempered representations in $\prod^M_{\phi_0}$ and the positive quasi-character ν_+ of $M(\mathbb{R})$. On the other hand $C^M_{\phi_0}$ equals C^G_{ϕ}, so we can define the pairing on $C^G_{\phi} \times \prod^G_{\phi}$ to be the one obtained from the pairing on $C^M_{\phi_0} \times \prod^M_{\phi_0}$.

However, simply defining the pairing for nontempered ϕ is not satisfactory. For it could well happen that the distribution

$$\sum_{\pi \in \prod_{\phi}} \mathrm{tr}(\pi)$$

is not stable if the parameter ϕ is not tempered. A related difficulty is that (1.2.3) no longer makes sense if ϕ_1 is not a tempered parameter for H. We shall define a subset of $\Phi(G/\mathbb{R})$ for which these difficulties are likely to have nice solutions. The subset will contain $\Phi_{\mathrm{temp}}(G/\mathbb{R})$, and ought also to account for the representations of $G(\mathbb{R})$ which are of interest in global applications.

Let $\Psi(G/\mathbb{R})$ be the set of ${}^LG^0$-conjugacy classes of maps

$$\psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L G$$

such that the restriction of ψ to $W_{\mathbb{R}}$ belongs to $\Phi_{temp}(G/\mathbb{R})$. For any $\psi \in \Psi(G/\mathbb{R})$ define a parameter ϕ_ψ in $\Phi(G/\mathbb{R})$ by

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \qquad w \in W_{\mathbb{R}}.$$

Here it is helpful to recall that

$$w \to \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}$$

is the map from $W_{\mathbb{R}}$ to

$$SL(2,\mathbb{C}) = {}^L(PGL(2))^0$$

which assigns the trivial representation to $PGL(2,\mathbb{R})$. Recall also that the unipotent conjugacy classes in any complex group are bijective with the conjugacy classes of maps of $SL(2,\mathbb{C})$ into the group. The unipotent conjugacy classes for complex groups have been classified by weighted Dynkin diagrams. (See [13].) Now any $\psi \in \Psi(G/\mathbb{R})$ can be identified with a pair (ϕ,ρ), in which $\phi \in \Phi_{temp}(G/\mathbb{R})$ and ρ is a map from $SL(2,\mathbb{C})$ into C_ϕ, given up to conjugacy by C_ϕ. From the classification of nilpotents it follows that ρ is determined by its restriction to the diagonal subgroup of $SL(2,\mathbb{C})$. We obtain

<u>Proposition 1.3.1</u>: The map

$$\psi \to \phi_\psi, \qquad \psi \in \Psi(G/\mathbb{R}),$$

is an injection from $\Psi(G/\mathbb{R})$ into $\Phi(G/\mathbb{R})$.

Thus, $\Psi(G/\mathbb{R})$ can be regarded as a subset of $\Phi(G/\mathbb{R})$. It contains $\Phi_{temp}(G/\mathbb{R})$ as the set of $\psi = (\phi,\rho)$ with ρ trivial.

Conjecture 1.3.2: For any $\psi \in \Psi(G/\mathbb{R})$, the representations in Π_{ϕ_ψ} are all unitary.

Suppose that $\psi = (\phi,\rho)$ is an arbitrary parameter in $\Phi(G/\mathbb{R})$. Copying a previous definition we set

$$C_\psi = C_\psi^G = \mathrm{Cent}(\psi(W_{\mathbb{R}} \times SL(2,\mathbb{C})), {}^LG^0)$$

and

$$\mathcal{C}_\psi = \mathcal{C}_\psi^G = C_\psi / C_\psi^0 Z_G .$$

The group C_ψ always equals $\mathrm{Cent}(\rho(SL(2,\mathbb{C}), C_\phi))$, and in particular is contained in C_ϕ. Therefore, there are natural maps $C_\psi \to C_\phi$ and $\mathcal{C}_\psi \to \mathcal{C}_\phi$. It is easy to check that this second map is surjective. In other words, there is an injective map

$$\hat{\mathcal{C}}_{\phi_\psi} \to \hat{\mathcal{C}}_\psi$$

from the (irreducible) characters on $\mathcal{C}_{\phi_\psi}$ to the irreducible characters on $\mathcal{C}_\psi$.

Fix $\psi \in \Psi(G/\mathbb{R})$. Take one of the pairings $\langle\, , \rangle$ on $\mathcal{C}_{\phi_\psi} \times \Pi_{\phi_\psi}$ discussed above, as well as the associated function c on the conjugacy classes of C_{ϕ_ψ}/Z_G. We pull back c to a function on the conjugacy classes of C_ψ / Z_G. We conjecture that the set Π_{ϕ_ψ} can be enlarged and the pairing extended so that all the theory for tempered parameters holds in this more general setting.

Conjecture 1.3.3: There is a finite set Π_ψ of irreducible representations of $G(\mathbb{R})$ which contains Π_{ϕ_ψ}, a function

$$\varepsilon_\psi : \Pi_\psi \to \{\pm 1\}$$

which equals 1 on Π_{ϕ_ψ}, and an injective map

$$\pi \to \langle\cdot, \pi\rangle, \qquad\qquad \pi \in \Pi_\psi,$$

from Π_ψ into $\hat{C}_\psi$, all uniquely determined, with the following properties.

(i) π belongs to the subset Π_{ϕ_ψ} of Π_ψ if and only if the function $\langle\cdot,\pi\rangle$ lies in the image of $\hat{C}_{\phi_\psi}$ in C_ψ.

(ii) The invariant distribution

$$\sum_{\pi \in \Pi_\psi} \varepsilon_\psi(\pi) \langle 1,\pi\rangle \operatorname{tr}(\pi) \tag{1.3.4}$$

is stable. (If C_ψ is abelian, which is certainly the case most of the time, the distribution is

$$\sum_{\pi \in \Pi_\psi} \varepsilon_\psi(\pi) \operatorname{tr}(\pi),$$

which except for the signs $\varepsilon_\psi(\pi)$ is just the sum of the characters in the packet Π_ψ.) We shall denote the value of this distribution on the function (1.1.2) by $F(\psi)$.

(iii) Let s be a semisimple element in C_ψ/Z_G. Let $H = H_s$ be the unique endoscopic group such that

$$\psi(W_{\mathbb{R}} \times SL(2,\mathbb{C})) \subset {}^L H \subset {}^L G,$$

so that, in particular, ψ defines a parameter in $\Psi(H/\mathbb{R})$. Then if $f \in C_c^\infty(G(\mathbb{R}))$, and $\bar{s}$ is the image of s in C_ψ,

$$f_H(\psi) = c(s) \sum_{\pi \in \Pi_\psi} \varepsilon_\psi(\pi) \langle \bar{s},\pi\rangle \operatorname{tr} \pi(f).$$

It is not hard to check the uniqueness assertion of this conjecture. The third condition states that

$$\hat{\chi}_{(\psi,x)}(f) = c(s)^{-1} f_{H_s}(\psi)$$

depends only on the projection x of s onto C_ψ, and that for any irreducible character Θ in $\hat{C}_\psi$,

$$(1.3.5) \qquad \frac{1}{|C_\psi|} \sum_{x \in C_\psi} \hat{\chi}_{(\psi,x)}(f)\overline{\Theta(x)} = \begin{cases} \varepsilon_\psi(\pi)\operatorname{tr}\pi(f), \text{ if } \Theta = \langle\cdot,\pi\rangle \text{ for some } \pi \in \Pi_\psi, \\ 0, \text{ otherwise}. \end{cases}$$

Assume inductively that the distribution (1.3.4) has been defined and shown to be stable whenever G is replaced by a proper endoscopic group $H = H_s$. Since the function f_{H_s} has already been defined on any stable distribution, the numbers $f_{H_s}(\psi)$ and $\hat{\chi}_{(\psi,x)}(f)$, with $\bar{s} = x \neq 1$, then make sense. To define $f_G(\psi)$, take $\Theta = 1$. If π_1 is the representation in Π_{ϕ_ψ} such that $\langle\cdot,\pi_1\rangle$ equals 1, we obtain

$$(1.3.6) \qquad |C_\psi|\operatorname{tr}\pi_1(f) = \sum_{x\in C_\psi} \hat{\chi}_{(\psi,x)}(f) .$$

The distribution

$$f_G(\psi) = \hat{\chi}_{(\psi,1)}(f)$$

is then equal to

$$(1.3.7) \qquad |C_\psi|\operatorname{tr}\pi_1(f) - \sum_{\substack{x\in C_\psi \\ x\neq 1}} \hat{\chi}_{(\psi,x)}(f) .$$

To complete the inductive definition, it is necessary to show it is stable. The formula (3.1.5) would then give the elements in Π_ψ uniquely, but only as <u>virtual characters</u>. The remaining problem is to show that the nonzero elements among them are linearly independent, and that up to a sign (which would serve as the definition of ε_ψ) they are irreducible characters.

The packets Π_ψ should have some other nice properties. For example, one can associate an R-group to any $\psi \in \Psi(G/\mathbb{R})$. Define

R_ψ to be the quotient of C_ψ by the group of components in C_ψ/Z_G which act on the identity component by inner automorphisms. If R_ψ is not trivial, the identity component will also not be trivial. The image of ψ will be contained in a Levi component of a proper parabolic subgroup of ${}^L G$. Let ${}^L M$ be a minimal Levi subgroup of ${}^L G$ which contains the image of ψ. Then ψ also represents a parameter in $\Psi(M/\mathbb{R})$. There is a short exact sequence

$$1 \to C_\psi^M \to C_\psi^G \to R_\psi \to 1.$$

The group R_ψ should govern the reducibility of the induced representations

$$\rho_\sigma = \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} (\sigma \otimes id_N), \qquad \sigma \in \Pi_\psi^M,$$

where $P = MN$ is a parabolic subgroup of G. Note that ρ_σ is obtained by unitary induction from a representation which is in general not tempered.

Finally, the conjecture should admit extensions in two directions - to real groups which are not necessarily quasi-split, and to pairs (G,α), where α is an automorphism of G (modulo the group of inner automorphisms). Both will eventually be needed to exploit the trace formula in full generality.

1.4. Conjecture 1.3.3 is suggested by the global situation, which we will come to later. I do not have much local evidence. The largest group for which I have been able to verify the conjecture completely is $PSp(4)$. However, even this group is instructive. We shall look at three examples which illustrate why it is the parameters ψ, and not ϕ_ψ, which govern questions of stability of characters. In each case, Π_{ϕ_ψ} will consist of one representation π such that $tr(\pi)$ is not stable. However, each group C_ψ will be of order two, and the sets $\prod_\psi$ will consist of π and another representation. It is only with these larger sets that we obtain a nice theory of stability.

In each example we will consider parameters ψ for $G = PSp(4)$ such that the projection of ψ onto ${}^LG^0$ factors through the endoscopic group

$$ {}^LH^{0'} = {}^LH^0_s \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C}), $$

with

$$ s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. $$

As we have said, $\prod_{\phi_\psi}$ will consist of one representation π. It will be the Langlands quotient of a nonunitarily induced representation ρ of $G(\mathbb{R})$. We shall let π^H denote the unique representation in the packet $\prod^H_{\phi_\psi} = \prod^H_\psi$, and we shall let ρ^H be the nonunitarily induced representation of $H(\mathbb{R})$ of which π^H is the Langlands quotient.

In order to deal with Ψ-parameters on $G(\mathbb{R})$, we must first know something about the Φ-parameters. The L-packets

$$ \prod_\phi, \qquad \phi \in \Phi(G/\mathbb{R}), $$

contain one or two elements. Those with two elements contain discrete series or limits of discrete series. They are of the form

$$ \prod_\phi = \{\pi_{Wh}, \pi_{hol}\}, $$

where π_{Wh} has a Whitaker model, and π_{hol} is the irreducible representation of $PSp(4,\mathbb{R})$ which combines the holomorphic and anti-holomorphic (limits of) discrete series for $Sp(4,\mathbb{R})$. We take the pairing $\langle , \rangle$ on $C_\phi \times \prod_\phi$ so that $\langle \cdot, \pi_{Wh}\rangle$ is the trivial character on $C_\phi \cong \mathbb{Z}/2\mathbb{Z}$, and $\langle \cdot, \pi_{hol}\rangle$ is the nontrivial character. It is not hard to verify that with this choice of pairing, all Shelstad's functions $c(s)$ may be taken to be 1. In our examples, we shall consider only representations with singular infinitesimal character, since these are the most difficult to handle. For this reason, $\{\pi_{Wh}, \pi_{hol}\}$ will now denote the L-packet in $G(\mathbb{R})$ which contains the lowest limits of discrete series. If π^H_{disc} is the lowest discrete series for $H(\mathbb{R})$,

$$\operatorname{tr} \pi^H_{disc}(f_H) = \operatorname{tr} \pi_{Wh}(f) - \operatorname{tr} \pi_{hol}(f) ,$$

for any $f \in C_c^\infty(G(\mathbb{R}))$. On the other hand, it will be clear in each example that

$$\operatorname{tr} \rho^H(f_H) = \operatorname{tr} \rho(f) ,$$

with ρ and ρ^H as above. As a distribution on $G(\mathbb{R})$, this last expression is stable.

We will prove the conjecture in each example by looking at the expression (1.3.7) for $f_G(\psi)$. If π is the unique representation in $\prod_{\phi_\psi}$, it will equal

$$2 \operatorname{tr} \pi(f) - f_H(\psi) .$$

To check the stability of this distribution, we will need to express it as a linear combination of <u>standard</u> characters on $G(\mathbb{R})$. To then construct the packet $\prod_\psi$, we will have to rewrite the expression as a linear combination of <u>irreducible</u> characters. The term $2 \operatorname{tr} \pi(f)$ is handled by computing the character formula (1.1.3) for the representation π of $G(\mathbb{R})$. This can be accomplished by reducing to the case of regular infinitesimal character through the procedure in [12] and then using Vogan's algorithm obtained from the Kazdan-Lusztig conjectures [15]. We will only quote the answer. To deal with $f_H(\psi)$, we shall first write the character formula (1.1.3) for the representation π^H of $H(\mathbb{R})$. Since $H(\mathbb{R})$ is isomorphic to $PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$, such formulas are well known. We will then lift the resulting standard characters on $H(\mathbb{R})$ to characters on $G(\mathbb{R})$ using the remarks above.

<u>Example 1.4.1</u>: Let ψ be given by the diagram

$$\begin{tikzcd}
W_{\mathbb{R}} \arrow[d] \arrow[dr, "\mu"] & \times & SL(2,\mathbb{C}) \arrow[d, "id"] \\
SL(2,\mathbb{C}) & \times & SL(2,\mathbb{C}) \cong {}^{L}H^{0}
\end{tikzcd} ,$$

in which the vertical arrow on the left is the parameter for $PGL(2,\mathbb{R})$ which corresponds to the lowest discrete series, and the image of μ in $SL(2,\mathbb{C})$ is contained in $\{\pm 1\}$. The centralizers are given as follows.

C_{ϕ_ψ}	C_{ϕ_ψ}	C_ψ	C_ψ
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^{x}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

We write π_μ for the representation in $\prod_{\phi_\psi}$. As we have agreed, ρ_μ then denotes the standard representation of which π_μ is the quotient, and π_μ^H and ρ_μ^H denote the corresponding representations of $H(\mathbb{R})$. The character formula (1.1.3) is easily shown to be

$$\operatorname{tr} \pi_\mu(f) = \operatorname{tr} \rho_\mu(f) - \operatorname{tr} \pi_{Wh}(f) .$$

On the other hand, from the well known character formula for π_μ^H we obtain

$$\begin{aligned}
f_H(\psi) &= \operatorname{tr} \pi_\mu^H(f_H) \\
&= \operatorname{tr} \rho_\mu^H(f_H) - \operatorname{tr} \pi_{disc}^H(f_H) \\
&= \operatorname{tr} \rho_\mu(f) - \operatorname{tr} \pi_{Wh}(f) + \operatorname{tr} \pi_{hol}(f) .
\end{aligned}$$

From our formula for $\operatorname{tr} \pi_\mu(f)$ we see that this equals

$$\operatorname{tr} \pi_\mu(f) + \operatorname{tr} \pi_{hol}(f).$$

Thus, the distribution

$$f_G(\psi) = 2 \operatorname{tr} \pi_\mu(f) - f_H(\psi)$$

on one hand equals

$$\operatorname{tr} \pi_\mu(f) - \operatorname{tr} \pi_{hol}(f) ,$$

but can also be written as

$$\operatorname{tr} \rho_\mu(f) - (\operatorname{tr} \pi_{Wh}(f) + \operatorname{tr} \pi_{hol}(f)) .$$

From the second expression we see that it is stable. From the first expression we see that the other assertions of the conjecture hold if we define

$$\Pi_\psi = \{\pi_\mu, \pi_{hol}\} ,$$

$$\varepsilon_\psi(\pi_\mu) = 1, \quad \varepsilon_\psi(\pi_{hol}) = -1 ,$$

and

$$\langle \cdot, \pi_\mu \rangle = 1, \quad \langle \cdot, \pi_{hol} \rangle = -1 .$$

We could have defined ψ so that the vertical arrow on the left corresponded to a higher discrete series of $PGL(2,\mathbb{R})$. Everything would have been the same except that $\{\pi_{Wh}, \pi_{hol}\}$ would stand for a pair of discrete series of $G(\mathbb{R})$. These examples are the local analogues of the nontempered cusp forms of $PSp(4)$ discovered by Kurakawa [7]. (See also [9(d), §3].)

<u>Example 1.4.2</u>: Define ψ by the diagram

$$\begin{tikzcd}
W_{\mathbb{R}} \arrow[d, "\mu_1"'] \arrow[drr, "\mu_2" near start] & \times & SL(2,\mathbb{C}) \arrow[dll, "id" near start] \arrow[d, "id"] \\
SL(2,\mathbb{C}) & \times & SL(2,\mathbb{C}) \cong L_H{}^0 ,
\end{tikzcd}$$

in which the images of μ_1 and μ_2 are contained in $\{\pm 1\}$, and $\mu_1 \neq \mu_2$. The centralizers are

C_{ϕ_ψ}	C_{ϕ_ψ}	C_ψ	C_ψ
$\mathbb{C}^\times \times \mathbb{C}^\times$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

We write π_{μ_1,μ_2} for the representation in $\prod_{\phi_\psi}$, and follow the notation above. The character formula (1.1.3) can be calculated to be

$$\operatorname{tr} \pi_{\mu_1,\mu_2}(f) = \operatorname{tr} \rho_{\mu_1,\mu_2}(f) - \operatorname{tr} \rho_{\mu_1}(f) - \operatorname{tr} \rho_{\mu_2}(f) + \operatorname{tr} \pi_{Wh}(f) .$$

On the other hand, from the character formula for $\pi^H_{\mu_1,\mu_2}$ we obtain

$$\begin{aligned}
f_H(\psi) &= \operatorname{tr} \pi^H_{\mu_1,\mu_2}(f_H) \\
&= \operatorname{tr} \rho^H_{\mu_1,\mu_2}(f_H) - \operatorname{tr} \rho^H_{\mu_1}(f_H) - \operatorname{tr} \rho^H_{\mu_2}(f_H) + \operatorname{tr} \pi^H_{disc}(f_H) \\
&= \operatorname{tr} \rho_{\mu_1,\mu_2}(f) - \operatorname{tr} \rho_{\mu_1}(f) - \operatorname{tr} \rho_{\mu_2}(f) + \operatorname{tr} \pi_{Wh}(f) - \operatorname{tr} \pi_{hol}(f) \\
&= \operatorname{tr} \pi_{\mu_1,\mu_2}(f) - \operatorname{tr} \pi_{hol}(f) .
\end{aligned}$$

Thus, the distribution

$$f_G(\psi) = 2 \operatorname{tr} \pi_{\mu_1,\mu_2}(f) - f_H(\psi)$$

on one hand equals

$$\mathrm{tr}\ \pi_{\mu_1,\mu_2}(f) + \mathrm{tr}\ \pi_{hol}(f) ,$$

but can also be written as

$$\mathrm{tr}\ \rho_{\mu_1,\mu_2}(f) - \mathrm{tr}\ \rho_{\mu_1}(f) - \mathrm{tr}\ \rho_{\mu_2}(f) + (\mathrm{tr}\ \pi_{Wh}(f) + \mathrm{tr}\ \pi_{hol}(f)) .$$

From the second expression we see that it is stable. From the first expression we obtain the other assertions of the conjecture if we define

$$\Pi_\psi = \{\pi_{\mu_1,\mu_2}, \pi_{hol}\} ,$$

$$\varepsilon_\psi(\pi_{\mu_1,\mu_2}) = 1 = \varepsilon_\psi(\pi_{hol}) ,$$

and

$$\langle \cdot, \pi_{\mu_1,\mu_2}\rangle = 1, \qquad \langle \cdot, \pi_{hol}\rangle = -1 .$$

This example is the local analogue of the nontempered cusp forms discovered by Howe and Piatetski-Shapiro [3].

Example 1.4.3: Define ψ as in the last example, except now take $\mu_1 = \mu_2 = \mu$. This example is perhaps the most striking. It is different from the previous two in that ψ factors through a Levi subgroup ${}^LM$ of a proper parabolic subgroup of ${}^LG$. (It is the maximal parabolic subgroup ${}^LP = {}^LM\,{}^LN$ whose unipotent radical is abelian.) This shows up in the fact that C_ψ is infinite.

C_{ϕ_ψ}	C_{ϕ_ψ}	C_ψ	C_ψ
$GL(2,\mathbb{C})$	$\{1\}$	$O(2,\mathbb{C})$	$\mathbb{Z}/2\mathbb{Z}$

We write $\pi_{\mu,\mu}$ for the representation in $\prod_{\phi_\psi}$, and follow the notation above. The character formula for $\pi_{\mu,\mu}$ is the most complicated of the three to compute. It is

$$\mathrm{tr}\ \pi_{\mu,\mu}(f) = \mathrm{tr}\ \rho_{\mu,\mu}(f) - \mathrm{tr}\ \rho_\mu(f) - \mathrm{tr}\ \pi_{hol}(f) .$$

We also have

$$\begin{aligned} f_H(\psi) &= \mathrm{tr}\ \pi^H_{\mu,\mu}(f_H) \\ &= \mathrm{tr}\ \rho^H_{\mu,\mu}(f_H) - 2\ \mathrm{tr}\ \rho^H_\mu(f_H) + \mathrm{tr}\ \rho^H_{disc}(f) \\ &= \mathrm{tr}\ \rho_{\mu,\mu}(f) - 2\ \mathrm{tr}\ \rho_\mu(f) + \mathrm{tr}\ \pi_{Wh}(f) - \mathrm{tr}\ \pi_{hol}(f) . \end{aligned}$$

From our formula for $\mathrm{tr}\ \pi_{\mu,\mu}(f)$ and the formula for $\mathrm{tr}\ \pi_\mu(f)$ in Example 1.4.1, we see that this equals

$$\mathrm{tr}\ \pi_{\mu,\mu}(f) - \mathrm{tr}\ \pi_\mu(f) .$$

Thus, the distribution

$$f_G(\psi) = 2\ \mathrm{tr}\ \pi_{\mu,\mu}(f) - f_H(\psi)$$

on one hand equals

$$\mathrm{tr}\ \pi_{\mu,\mu}(f) + \mathrm{tr}\ \pi_\mu(f) ,$$

but can also be written as

$$\mathrm{tr}\ \rho_{\mu,\mu}(f) - (\mathrm{tr}\ \pi_{Wh}(f) + \mathrm{tr}\ \pi_{hol}(f)) .$$

From the second expression we see that it is stable. From the first expression we obtain the other assertions of the conjecture for the endoscopic groups G and H if we define

$$\prod_\psi = \{\pi_{\mu,\mu}, \pi_\mu\} ,$$

$$\varepsilon_\psi(\pi_{\mu,\mu}) = 1 = \varepsilon_\psi(\pi_\mu) ,$$

and

$$\langle\cdot,\pi_{\mu,\mu}\rangle = 1, \qquad \langle\cdot,\pi_\mu\rangle = -1 .$$

In this example we have a third endoscopic group to consider - the Levi subgroup M, which we can identify with $GL(2)$. Since ψ factors through ${}^LM$, it defines a parameter in $\Psi(M/\mathbb{R})$. To complete the verification of the conjecture we must show that

$$f_M(\psi) = \operatorname{tr}\pi_{\mu,\mu}(f) + \operatorname{tr}\pi_\mu(f) .$$

The packets $\prod_{\phi_\psi}^M$ and $\prod_\psi^M$ both consist of one element, the representation

$$\sigma(m) = \mu(\det(m)), \qquad m \in GL(2,\mathbb{R}).$$

The definitions of Shelstad are set up so that the map

$$f \to f_M$$

is dual to induction. Therefore, we will be done if we can show that the induced representation

$$\rho_\sigma = \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\sigma \otimes \mathrm{id}_N)$$

is the direct sum of $\pi_{\mu,\mu}$ and π_μ. Now, σ is a nontempered unitary character of $M(\mathbb{R})$. It is the difference between a nontempered standard character on $GL(2,\mathbb{R})$ and a lowest discrete series on $GL(2,\mathbb{R})$. The induced character $\operatorname{tr}(\rho_\sigma)$ is the difference between the corresponding two induced standard characters. The first is just $\operatorname{tr}(\rho_{\mu,\mu})$. The second is a tempered character on $G(\mathbb{R})$ which is reducible; its constituents are π_{Wh} and π_{hol}. Therefore, our induced character equals

$$\mathrm{tr}(\rho_{\mu,\mu}) - (\mathrm{tr}(\pi_{Wh}) + \mathrm{tr}(\pi_{hol})) ,$$

which, as we have seen above, is just

$$\mathrm{tr}(\pi_{\mu,\mu}) + \mathrm{tr}(\pi_{\mu}) .$$

It follows that

$$\rho_{\sigma} = \pi_{\mu,\mu} \oplus \pi_{\mu} ,$$

as required.

Notice that $C_{\psi} \cong O(2,\mathbb{C})$ acts on $C_{\psi}^{0} \cong SO(2,\mathbb{C})$ by outer automorphism. Consequently,

$$R_{\psi} = C_{\psi} \cong \mathbb{Z}/2\mathbb{Z} .$$

Therefore the order of the R group is equal to the number of irreducible constituents of the induced representation

$$\rho_{\sigma} = \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} (\sigma \otimes \mathrm{id}_{N}) ,$$

as we would hope. Observe that the analogue of the R group for the parameter ϕ_{ψ} is trivial. Thus, we see a further example of behaviour which is tied to the parameter ψ rather than ϕ_{ψ}.

This suggests a concrete problem.

<u>Problem 1.4.4</u>: Let $\mathfrak{a}_M$ be the Lie algebra of the split component of the center of $M(\mathbb{R})$. The Weyl group of $\mathfrak{a}_M$ is in this case isomorphic to R_{ψ}. Let w be a representative in $G(\mathbb{R})$ of its non-trivial element. It is known that the corresponding intertwining operator between $\rho_{\sigma_{\lambda}}$ and $\rho_{\sigma_{-\lambda}}$ can be normalized according to the prescription in [9(b),Appendix II]. Let $N(w)$ be the value of the normalized intertwining operator at $\lambda = 0$. It is a unitary operator whose square is 1. Its definition is canonical up to a choice of the

representative w in $G(\mathbb{R})$. The problem is to show that $N(w)$ is not a scalar, and more precisely, to show that if the determinant of w is positive, then

$$\begin{aligned} \operatorname{tr}(N(w)\rho_\sigma(f)) &= \operatorname{tr}\pi_{\mu,\mu}(f) - \operatorname{tr}\pi_\mu(f) \\ &= \sum_{\pi\in\Pi_\psi} \langle \bar{s},\pi\rangle \operatorname{tr}\pi(f) \quad . \end{aligned}$$

§2. A GLOBAL CONJECTURE

2.1. The conjecture we have just stated can be made for any local field F. If F is non-Archimedean, however, the Weil group must be replaced by the group

$$W_F' = W_F \times SL(2,\mathbb{C})$$

introduced in [9(d)]. If G is a reductive quasi-split group defined over F, $\Phi(G/F)$ must be taken to be the set of equivalence classes of maps

$$W_F \times SL(2,\mathbb{C}) \to {}^LG ,$$

while $\Phi_{temp}(G/F)$ will be the subset of those maps whose restriction to W_F has bounded image, when projected onto ${}^LG^0$. In order to define the parameters ψ we must add on another $SL(2,\mathbb{C})$. We take $\Psi(G/F)$ to be the set of ${}^LG^0$ conjugacy classes of maps

$$\psi\colon W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to {}^LG$$

such that the restriction of ψ to the product of W_F with the first $SL(2,\mathbb{C})$ belongs to $\Phi_{temp}(G/F)$. For any such ψ, the parameter

$$\phi_\psi(w,\sigma) = \psi\left(w,\sigma,\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2}\end{pmatrix}\right), \qquad w\in W_F,\ \sigma\in SL(2,\mathbb{C}),$$

belongs to $\Phi(G/F)$.

The conjecture also has a global analogue. Let F be a global field with adèle ring $\mathbb{A}$, and let G be a reductive group over F. If G is not split, there are minor complications in the definitions related to endoscopic groups. (See [9(e)].) To avoid discussing them we shall simply take G to be split. Then the global definitions connected with endoscopic groups follow exactly the local ones we have given.

The conjecture will describe the automorphic representations which are "tempered" in the global sense; that is, representations which occur in the direct integral decomposition of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$. However, we cannot use the global Weil group if we want to account for all such representations. For even $GL(2)$ has many cuspidal automorphic representations which will not be attached to two dimensional representations of the Weil group. The simplest way to state the global conjecture is to use the conjectural Tannaka group, discussed in [9(d)]. If certain properties hold for the representations of $GL(n)$, Langlands points out that there will be a complex, reductive pro-algebraic group $\mathcal{G}_{\Pi_{temp}(F)}$ whose n-dimensional (complex analytic) representations parametrize the automorphic representations of $GL(n,\mathbb{A})$ which are tempered at each place. For each place v, there will also be a complex, reductive pro-algebraic group $\mathcal{G}_{\Pi_{temp}(F_v)}$, equipped with a map

$$\mathcal{G}_{\Pi_{temp}(F_v)} \to \mathcal{G}_{\Pi_{temp}(F)} \quad ,$$

whose n-dimensional representations parametrize the tempered representations of $GL(n,F_v)$. The composition of this map with an n-dimensional representations of $\mathcal{G}_{\Pi_{temp}(F)}$ will give the F_v-constituent of the corresponding automorphic representation.

The sets $\Psi(G/F_v)$ which we have defined could also be described as the set of ${}^LG^0$ conjugacy classes of maps

$$\psi_v\colon \mathcal{G}_{\Pi_{temp}(F_v)} \times SL(2,\mathbb{C}) \to {}^LG$$

The centralizer in ${}^LG^0$ of the image of ψ_v is the same as the centralizer of the image of the corresponding parameter associated to the Weil group. In other words,

$$C_{\psi_v} = \mathrm{Cent}(\psi_v({}_G\Pi_{temp}(F_v) \times SL(2,\mathbb{C})), {}^LG^0)$$

and

$$\mathcal{C}_{\psi_v} = C_{\psi_v}/C^0_{\psi_v} Z_G .$$

We make the same definitions globally. Assuming the existence of the groups ${}_G\Pi_{temp}(F)$ and ${}_G\Pi_{temp}(F_v)$, let $\Psi(G/F)$ be the set of ${}^LG^0$ conjugacy classes of maps

$$\psi: {}_G\Pi_{temp}(F) \times SL(2,\mathbb{C}) \to {}^LG .$$

If $\psi \in \Psi(G/F)$ is any such global parameter, set

$$C_\psi = \mathrm{Cent}(\psi({}_G\Pi_{temp}(F) \times SL(2,\mathbb{C})) , {}^LG^0) ,$$

and

$$\mathcal{C}_\psi = C_\psi/C^0_\psi Z_G .$$

The composition of the map

$${}_G\Pi_{temp}(F_v) \times SL(2,\mathbb{C}) \to {}_G\Pi_{temp}(F) \times SL(2,\mathbb{C})$$

with ψ gives a parameter $\psi_v \in \Psi(G/F_v)$. There are natural maps

$$C_\psi \to C_{\psi_v}$$

and

$$C_\psi \to C_{\psi_v}$$

Assume that the analogue of the local Conjecture 1.3.3 holds for each field F_v. Fix $\psi \in \Psi(G/F)$. Then for any place v we have a finite set $\prod_{\psi_v}$, a function ε_{ψ_v} on $\prod_{\psi_v}$, a pairing

$$\langle x_v, \pi_v \rangle, \qquad \pi_v \in \textstyle\prod_{\psi_v},\ x_v \in C_{\psi_v} ,$$

and a function c_v on the conjugacy classes of C_{ψ_v}/Z_G. Define the global packet $\prod_\psi$ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each v, π_v belongs to $\prod_{\psi_v}$. Define the global pairing

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle$$

and the global function

$$\varepsilon_\psi(\pi) = \prod_v \varepsilon_{\psi_v}(\pi_v)$$

for $\pi = \otimes_v \pi_v$ in $\prod_\psi$ and x in C_ψ with image x_v in C_{ψ_v}. Almost all the terms in each product should equal 1. It is reasonable to expect that for any element $s \in C_\psi/Z_G$, with image s_v in C_{ψ_v}/Z_G,

$$\prod_v c_v(s_v) = 1.$$

If this is so, the global pairing will be canonical.

<u>Conjecture 2.1.1</u>: (A) The representations of $G(\mathbb{A})$ which occur in the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ occur in packets parametrized by $\Psi(G/F)$. The representations in the packet corresponding to ψ will occur in the discrete spectrum if and only if C_ψ is finite.

(B) Suppose that C_ψ is finite. Then there is a positive integer d_ψ and a homomorphism

$$\xi_\psi : C_\psi \to \{\pm 1\}$$

such that the multiplicity with which any $\pi \in \Pi_\psi$ occurs discretely in $L^2(G(F)\backslash G(\mathbb{A}))$ equals

$$\frac{d_\psi}{|C_\psi|} \sum_{x \in C_\psi} \langle x, \pi \rangle \; \xi_\psi(x).$$

In particular, if C_ψ and each C_{ψ_v} are abelian, the multiplicity of π is d_ψ if the character $\langle\cdot,\pi\rangle$ equals ξ_ψ, and is zero otherwise.

2.2. Some comments are in order. First of all, the introduction of the Tannaka groups would seem to put the conjecture on a rather shaky foundation. However, everything may be formulated without them. The set $\Psi(G/F)$ is the same as the collection of pairs (ϕ,ρ), where $\phi \in \Phi_{temp}(G/F)$ and ρ is a map from $SL(2,\mathbb{C})$ into C_ϕ, given up to conjugacy by C_ϕ. Included in the conjecture (and also implicit in [9(d)]) is the assertion that $\Phi_{temp}(G/F)$ is the set of L equivalence classes of automorphic representations of $G(\mathbb{A})$ which are tempered at every place. We could simply take this as the definition of $\Phi_{temp}(G/F)$. To avoid mentioning the Tannaka group at all, we would need to define C_ϕ for each ϕ in $\Phi_{temp}(G/F)$. For then C_ψ would just be the centralizer of the image of ρ in C_ϕ. If one grants the existence of certain liftings, one can show that C_ϕ is equal to the centralizer in ${}^LG^0$ of an embedded L-group in ${}^LG$.

Notice that the conjecture does not specify whether an automorphic representation which occurs in the discrete spectrum is cuspidal or not. Indeed, it is quite possible for a global packet Π_ψ to contain one representation which is cuspidal and another which occurs in the residual discrete spectrum. (See [2] and also Example 2.4.1 below.) I do not know whether there will be a simple explanation for such behaviour.

Multiplicity formulas of the sort we conjecture first appeared in [8]. The integer d_ψ was needed there, even for subgroups of $\mathrm{Res}_{E/F}(GL(2))$, to account for distinct global parameters which were everywhere locally equivalent. The sign characters ξ_ψ are more mysterious. Suppose that ${}^LG^0$ is the set of fixed points of an outer automorphism of $GL(n,\mathbb{C})$. Then one can observe the existence of such characters from the anticipated properties of the twisted trace formula for $GL(n)$. The character will be 1 if ψ corresponds to a pair (ϕ,ρ) with ρ trivial; that is, if the representations in Π_ψ are tempered at each local place. In general, however, ξ_ψ will not be trivial, and will be built out of the orders at 1/2 of certain L-functions of ϕ. Incidentally, in the examples I have looked at, both local and global, the groups C_ψ have all been abelian. The extrapolation to nonabelian C_ψ is no more than a guess. In fact if C_ψ is nonabelian, the functions $\langle\ ,\pi\rangle$ may turn out to be only class functions on C_ψ, and not irreducible characters."

2.3. Let us look at a few examples. Consider first the group $G=GL(n)$. The centralizer of any reductive subgroup of ${}^LG^0=GL(n,\mathbb{C})$ is connected. This means that the packet Π_ψ (both local and global) should each contain only one representation. The groups C_ϕ will be of the form

$$GL(n_1,\mathbb{C}) \times \dots \times GL(n_r,\mathbb{C}) ,$$

so that a parameter ψ will consist of the tempered parameter ϕ and a map of $SL(2,\mathbb{C})$ into this group. The representations in Π_ψ should belong to the discrete spectrum (modulo the center of $G(\mathbb{A})$) if and only if C_ψ equals $\mathbb{C}^\times$. This will be the case precisely when C_ϕ equals $GL(n_1,\mathbb{C})$ and ρ is the irreducible n_1 dimensional representation of $SL(2,\mathbb{C})$. Then n_1 will necessarily divide n, $n = n_1 m$, and ϕ will be identified with a cuspidal automorphic representation of $GL(m,\mathbb{A})$, enbedded diagonally in $GL(n)$. This prescription for the discrete spectrum of $GL(n,\mathbb{A})$ (modulo the center) is exactly what is expected. (See [4].) It is only for $GL(n)$

(and closely related groups such as $SL(n)$) that the distinction between the cuspidal spectrum and the residual discrete spectrum will be so clear.

The multiplicity formula of the conjecture is compatible with the results of Labesse and Langlands [8] for $SL(2)$. More recently, Flicker [2] has studied the quasi-split unitary group in three variables. The conjecture, or rather its analogue for non-split groups, is compatible with his results.

Langlands has shows [9(b), Appendix 3] that for the split group G of type G_2 there is an interesting automorphic representation which occurs in the discrete noncuspidal spectrum. Its Archimedean component is infinite dimensional, of class one and is not tempered. The existence of such a representation is predicted by our conjecture. ${}^LG^0$ is just the complex group of type G_2. It has three unipotent conjugacy classes which meet no proper Levi subgroup. These correspond to the principal unipotent classes of the embedded subgroups

$$ {}^LH_i^0 \rightarrow {}^LG^0 \qquad\qquad i = 1,2,3, $$

where

$$ {}^LH_1^0 = {}^LG^0 $$

$$ {}^LH_2^0 \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\{\pm 1\} , $$

and

$$ {}^LH_3^0 \cong SL(3,\mathbb{C}) . $$

Let $\psi_i = (\phi, \rho_i)$ be the parameter in $\Psi(G/F)$ such that ϕ is trivial and ρ_i is the composition

$$ SL(2,\mathbb{C}) \rightarrow {}^LH_i^0 \rightarrow {}^LG^0 , $$

in which the map on the left is the one which corresponds to the principal unipotent class in ${}^L H_i^0$. The packet Π_{ψ_1} contains one element, the trivial representation of $G(\mathbb{A})$. It is the packet Π_{ψ_2} which should contain the representation discovered by Langlands. The remaining representations in Π_{ψ_2} which occur in the discrete spectrum, as well as all such representations in Π_{ψ_3}, are presumably cuspidal.

2.4. Finally, consider the global analogues for $PSp(4)$ of the three examples we discussed in §1. The global conjecture cannot be proved yet for this group, for there remain unsolved local problems. However, Piatetski-Shapiro has proved the multiplicity formulas of the first two examples below by different methods. (See [10(a)], [10(b)], [10(c)].) Using L-functions and the Weil representation, he reduced the proof to a problem which had been solved by Waldspurger [16].

In each example ψ will be given by the diagram for the corresponding local example in §1 except that $W_{\mathbb{R}}$ is to be replaced by the Tannaka group $G_{\Pi_{temp}(F)}$ or, as suffices in these examples, by the global Weil group W_F. Each μ will be a Grössencharacter of order 1 or 2, since the one dimensional representations of $G_{\Pi_{temp}(F)}$, W_F and $F^x \backslash \mathbb{A}^x$ all co-incide. In each example the integer d_ψ will be 1.

<u>Example 2.4.1</u>: This is the example of Kurakawa. Take the diagram in Example 1.4.1, letting the vertical arrow on the left parametrize a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of $PGL(2, \mathbb{A})$. As in the local case, we have

$$C_\psi \cong \mathbb{Z}/2\mathbb{Z}, \qquad C_\psi \cong \mathbb{Z}/2\mathbb{Z} .$$

The character ξ_ψ should be 1 or -1 according to whether the order at $s = 1/2$ of the standard L function $L(s,\tau)$ is even or odd. Our conjecture states that a representation π in the packet Π_ψ occurs in the discrete spectrum if and only if the character $\langle\pi,\cdot\rangle$ on C_ψ equals ξ_ψ. The local centralizer group C_{ψ_v} will be of order 2 or 1 depending on whether the representation τ_v of $PGL(2,F_v)$ belongs to the local discrete series or not. Suppose that τ_v belongs to the local discrete series at r different places. Then the global pocket Π_ψ will contain 2^r representations. Exactly half of them will occur in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. (If $r = 0$, the one representation in Π_ψ will occur in the discrete spectrum if and only if $\xi_\psi = 1$.)

For a given complex number s, consider the representation

$$(x,a) \to \tau(x)\mu(a)|a|^{\frac{s}{2}}, \qquad x \in PGL(2,\mathbb{A}),\ a \in \mathbb{A}^x ,$$

of $PGL(2,\mathbb{A}) \times \mathbb{A}^x$. It is an automorphic representation of a Levi subgroup of G which is cuspidal modulo the center. The associated induced representation of $G(\mathbb{A})$ will have a global intertwining operator, for which we can anticipate a global normalizing factor equal to

$$(L(\tfrac{s}{2},\tau)L(s,1_F))(L(-\tfrac{s}{2},\tau)L(-s,1_F))^{-1} .$$

From the theory of Eisenstein series and the expected properties of the local normalized intertwining operators, one can show that Π_ψ will have a representation in the residual discrete spectrum if and only if the function above has a pole at $s = 1$. This will be the case precisely when $L(1/2,\tau)$ does not vanish. Thus, the number of cuspidal automorphic representations in the packet Π_ψ should equal 2^{r-1} or $2^{r-1} - 1$, depending on whether $L(1/2,\tau)$ vanishes or not.

<u>Example 2.4.2</u>: This is the example of Howe and Piatetski-Shapiro. Take the diagram in Example 1.4.2 with $\mu_1 \neq \mu_2$. Then

$$C_\psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ , \qquad C_\psi \cong \mathbb{Z}/2\mathbb{Z}\ .$$

The character ξ_ψ should always be 1. Our conjecture states that a representation $\pi \in \prod_\psi$ will occur in the discrete spectrum if and only if the character $\langle \cdot , \pi \rangle$ equals 1. Each local centralizer group C_{ψ_v} will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It follows that the packet $\prod_\psi$ will contain infinitely many representations, and infinitely many should occur discretely in $L^2(G(F)\backslash G(\mathbb{A}))$.

<u>Example 2.4.3</u>: Take the diagram in Example 1.4.2 with $\mu_1 = \mu_2$. Then

$$C_\psi \cong O(2,\mathbb{C}), \qquad C_\psi \cong \mathbb{Z}/2\mathbb{Z}\ .$$

Each local centralizer group C_{ψ_v} will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so the packet $\prod_\psi$ will contain infinitely many representations. However, since C_ψ is infinite, the conjecture states that none of them will occur discretely in $L^2(G(F)\backslash G(\mathbb{A}))$.

§3. THE TRACE FORMULA

3.1. The conjecture of §2 can be motivated by the trace formula, if one is willing to grant the solutions of several local problems. We hope to do this properly on some future occasion, but at the moment even this is too large a task. We shall be content here to discuss a few problems connected with the trace formula, and to relate them to the conjecture in the example we have been looking at - the group $PSp(4)$. For a more detailed description of the trace formula, see the paper [1(b)] and the references listed there.

Let G be as in §2, but for simplicity, take F to be the field of rational numbers $\mathbb{Q}$. The trace formula can be regarded as an equality

$$\sum_{o \in O} I_o(f) = \sum_{\chi \in X} I_\chi(f), \qquad f \in C_c^\infty(G(\mathbb{A})), \tag{3.1.1}$$

of invariant distributions on $G(\mathbb{A})$. The distributions on the left are parametrized by the semisimple conjugacy classes in $G(\mathbb{Q})$, while those on the right are parametrized by cuspidal automorphic representations associated to Levi components of parabolic subgroups of G. Included in the terms on the left are orbital integrals on $G(\mathbb{A})$ (the distributions in which the semisimple conjugacy class in $G(\mathbb{Q})$ is regular elliptic) and on the right are the characters of cuspidal automorphic representations of $G(\mathbb{Q})$ (the distributions in which the Levi subgroup is G itself). In general the terms on the left are invariant distributions which are obtained naturally from <u>weighted</u> orbital integrals on $G(\mathbb{A})$. The terms on the right are simpler, and can be given by a reasonably simple explicit formula. (See [1(b)]).

The goal of [9(c)] was to begin an attack on a fundamental problem - to stabilize the trace formula. The endoscopic groups for G are quasi-split groups defined over $\mathbb{Q}$; they can be regarded as endoscopic groups over the completions $\mathbb{Q}_v$ of $\mathbb{Q}$. As in §1, we suppose that for each endoscopic group H we have fixed an admissible embedding ${}^LH \subset {}^LG$ which is compatible with equivalence. We also assume that the theory of Shelstad for real groups has been extended to an arbitrary local field. Then for any function $f \in C_c^\infty(G(\mathbb{A}))$ and any endoscopic group H we will be able to define a function f_H in $C_c^\infty(H(\mathbb{A}))$. For example, if f is of the form $\otimes_v f_v$, we simply set

$$f_H = \otimes_v f_{v,H}$$

However, f_H will be determined only up to evaluation on stable distributions on $H(\mathbb{A})$. To exploit the trace formula, it will be necessary to express the invariant distributions which occur in terms of stable distributions on the various groups $H(\mathbb{A})$.

Kottwitz [6] has introduced a natural equivalence relation, called <u>stable conjugacy</u>, on the set of conjugacy classes in $G(\overline{\mathbb{Q}})$ on the regular semisimple classes. If $\mathcal{O}$ is the set of all semisimple conjugacy classes in $G(\mathbb{Q})$, let $\overline{\mathcal{O}}$ be the set of stable conjugacy classes in $\mathcal{O}$. For any $\overline{\sigma} \in \overline{\mathcal{O}}$, set

$$I_{\overline{\sigma}}(f) = \sum_{\sigma \in \overline{\sigma}} I_{\sigma}(f), \qquad f \in C_c^{\infty}(G(\mathbb{A})) .$$

If H is an endoscopic group for G, it can be shown that there is a natural map

$$\overline{\mathcal{O}}_H \to \overline{\mathcal{O}}$$

from the semisimple stable conjugacy classes of $H(\mathbb{Q})$ to those of $G(\mathbb{Q})$. One of the main results of [9(e)] was a formula

$$(3.1.2) \qquad I_{\overline{\sigma}}(f) = \sum_H \iota(G,H) \sum_{\{\overline{\sigma}_H \in \overline{\mathcal{O}}_H : \overline{\sigma}_H \to \overline{\sigma}\}} S^H_{\overline{\sigma}_H}(f_H) ,$$

for any $f \in C_c^{\infty}(G(\mathbb{A}))$ and any class $\overline{\sigma} \in \overline{\mathcal{O}}$ consisting of regular elliptic elements. For each endoscopic group H, $\iota(G,H)$ is a constant and $S^H_{\overline{\sigma}_H}$ is a stable distribution on $H(\mathbb{A})$. The sum over H (as well as all such sums below) is taken over the equivalence classes of cuspidal endoscopic groups for G.

<u>Problem 3.1.3</u>: Show that the formula (3.1.2) holds for an arbitrary stable conjugacy class $\overline{\sigma}$ in $\overline{\mathcal{O}}$.

This problem is similar in spirit to that posed by Conjecture 1.3.3. It is not necessary to construct the stable distributions $S^H_{\overline{\sigma}_H}$. One would assume inductively that they had been defined for any $H \neq G$. (Of course we could not continue to work within the limited category we have adopted for this exposition - namely, G is a split group with embeddings ${}^L H \subset {}^L G$.) The problem would then amount to showing that the invariant distribution

$$f \to I_{\bar{\sigma}}(f) - \sum_{H \neq G} \iota(G,H) \sum_{\{\bar{\sigma}_H \to \bar{\sigma}\}} S^H_{\bar{\sigma}_H}(f_H)$$

was stable. However, this assertion is still likely to be quite difficult. The problem does not seem tractable, in general, without a good knowledge of the Fourier transforms of the distributions $I_{\bar{\sigma}}$.

In any case, assume Problem 3.1.3 has been solved. Define

$$I(f) = I^G(f) = \sum_{\bar{\sigma} \in \bar{\mathcal{O}}} I_{\bar{\sigma}}(f) ,$$

and

$$S(f) = S^G(f) = \sum_{\bar{\sigma} \in \bar{\mathcal{O}}} S^G_{\bar{\sigma}}(f) ,$$

for any $f \in C_c^\infty(G(\mathbb{A}))$. The expression for $I(f)$ is just equal to each side of the trace formula (3.1.1). It is clear that it converges absolutely. The same cannot be said of the expression for $S(f)$. The problem is discusses in [9(e),VIII.5]. We must make the assumption that there are only finitely many H such that $f_H \neq 0$. (See Lemma 8.12 of [9(e)].) This is certainly true if G is adjoint for then there are only finitely many endoscopic groups (up to equivalence, of course). Since the constant $\iota(G,G)$ equals 1, we obtain

$$\sum_{\bar{\sigma} \in \bar{\mathcal{O}}} S^G_{\bar{\sigma}}(f)$$

$$= \sum_{\bar{\sigma} \in \bar{\mathcal{O}}} \Big(I_{\bar{\sigma}}(f) - \sum_{H \neq G} \iota(G,H) \sum_{\{\bar{\sigma}_H \in \bar{\mathcal{O}}_H : \bar{\sigma}_H \to \bar{\sigma}\}} S^H_{\bar{\sigma}_H}(f_H)\Big)$$

$$= \sum_{\bar{\sigma}} I_{\bar{\sigma}}(f) - \sum_{H \neq G} \iota(G,H) \sum_{\bar{\sigma}_H \in \bar{\mathcal{O}}_H} S^H_{\bar{\sigma}_H}(f_H)$$

$$= I(f) - \sum_{H \neq G} \iota(G,H) S^H(f_H) ,$$

if we assume inductively that the expression used to define S^H converges absolutely whenever $H \neq G$. It follows that the expression for $S^G(f)$ converges absolutely, and S^G is a stable distribution on $G(\mathbb{A})$. Moreover,

$$I(f) = \sum_H \iota(G,H) S^H(f_H) , \tag{3.1.4}$$

for any $f \in C_c^\infty(G(\mathbb{A}))$.

3.2. An identity (3.1.4) could be used to yield interesting information about the discrete spectrum of G, since there is an explicit formula for

$$I(f) = \sum_{\chi \in \mathfrak{X}} I_\chi(f) . \tag{3.2.1}$$

The formula is given as a sum of integrals over vector spaces $i\mathfrak{a}_M^* / i\mathfrak{a}_G^*$, where $P = MN$ is a parabolic subgroup of G (defined over $\mathbb{Q}$), A_M is the split component of the center of the Levi component M of P, and $\mathfrak{a}_M$ is the Lie algebra of $A_M(\mathbb{R})$. The most interesting part of the formula is the term for which the integral is actually discrete; in other words, for which $P = G$. It is only this term that we shall describe.

Suppose that $P = MN$ is a parabolic subgroup and that σ is an irreducible unitary representation of $M(\mathbb{A})$. Let ρ_σ be the induced representation

$$\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} (L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))_\sigma \otimes \mathrm{id}_N) ,$$

where id_N is the trivial representation of the unipotent radical $N(\mathbb{A})$, and $L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))_\sigma$ is the σ-primary component of the subrepresentation of $M(\mathbb{A})$ on $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$ which decomposes discretely. Let $W(\mathfrak{a}_M)$ be the Weyl group of $\mathfrak{a}_M$, and let

$W(\mathfrak{a}_M)_{reg}$ be the subset of elements in $W(\mathfrak{a}_M)$ whose space of fixed vectors is $\mathfrak{a}_G$. For any w in $W(\mathfrak{a}_M)$ let $T(w)$ be the (unnormalized) global intertwining operator from ρ_σ to $\rho_{w\sigma}$. For any function $f \in C_c^\infty(G(\mathbb{A}))$, define

$$(3.2.2) \qquad I_+(f) = I_+^G(f)$$

$$= \sum_{\{(M,\sigma)\}} |W(\mathfrak{a}_M)|^{-1} \sum_{w \in W(\mathfrak{a}_M)_{reg}} |\det(1-w)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \operatorname{tr}(T(w)\rho_\sigma(f)) ,$$

where the first sum is over pairs (M,σ) as above, with M given up to $G(\mathbb{Q})$ conjugacy. Then I_+ is the "discrete part" of the explicit formula for (3.2.1). Here we have obscured a technical complication for the sake of simplicity. It is not known that the sum over σ in (3.2.2) converges absolutely (although one expects it to do so). In order to insure absolute convergence, one should really group the summands in (3.2.2) with other components of $I(f)$ in a way that takes account of the decomposition on the right hand side of (3.2.1).

We expect to be able to isolate the various contributions of (3.1.4) to the distribution I_+. This would mean that we could find (for every G) a stable distribution S_+^G on $G(\mathbb{A})$ such that

$$(3.2.3) \qquad I_+(f) = \sum_H \iota(G,H) S_+^H(f_H) ,$$

for any $f \in C_c^\infty(G(\mathbb{A}))$. Said another way, the distribution

$$f \to I_+(f) - \sum_{H \neq G} \iota(G,H) S_+^H(f_H) ,$$

would be stable. Now this is actually a rather concrete assertion. The distribution I_+ is certainly given by a concrete formula, and the distributions S_+^H are defined inductively in terms of the formulas for I_+^H. Moreover, Kottwitz has recently evaluated the constants

$\iota(G,H)$. We will not give the general formula, but if G and $H = H_s$ are both split groups, $\iota(G,H)$ equals

$$|Z_H/Z_G|^{-1}\ |\mathrm{Norm}(sZ_G, {}^LG^0)/{}^LH^0|^{-1}\ ,$$

where $\mathrm{Norm}(sZ_G, {}^LG^0)$ denotes the group of elements σ in ${}^LG^0$ which normalize the coset sZ_G.

A formula like (3.2.3) will have interesting implications for the discrete spectrum of G. Consider the one dimensional automorphic representations of the various endoscopic groups H. Our examples for $PSp(4,\mathbb{R})$ suggest that for $H \neq G$, the contributions of such one dimensional representations to the right hand side of (3.2.3) will not be stable distributions of f. They will have to correspond to something in the formula (3.2.2) for $I_+(f)$. Suppose that some one dimensional representations cannot be accounted for by any terms in (3.2.2) indexed by (M,σ), with $M \neq G$. Then they will have to correspond to terms with $M = G$. In other words, they ought to give rise to interesting nontempered automorphic representations of $G(\mathbb{A})$ which occur in the discrete spectrum.

It is implicit in our conjecture that we should index the one dimensional automorphic representations of $H(\mathbb{A})$ by maps

$$W_{\mathbb{Q}} \times SL(2,\mathbb{C}) \to {}^LH,$$

in which the image of $W_{\mathbb{Q}}$ in ${}^LH^0$ commutes with ${}^LH^0$ and the image of $SL(2,\mathbb{C})$ corresponds to the principal unipotent in ${}^LH^0$. (For the correspondence between unipotent conjugacy classes and representations of $SL(2,\mathbb{C})$, see [13].) It is of course easy to do this. What is not clear is why we should do it. Why introduce an $SL(2,\mathbb{C})$ when the one dimensional representations of $H(\mathbb{A})$ can be described perfectly well without it? According to the conjecture, the $SL(2,\mathbb{C})$ factor will be essential in describing the corresponding automorphic representations of $G(\mathbb{A})$. In particular, a one dimensional automorphic representation of $H(\mathbb{A})$ should give rise to automoprhic representations of $G(\mathbb{A})$ which occur discretely (modulo the center of $G(\mathbb{A})$) if and only if the image of $W_{\mathbb{Q}} \times SL(2,\mathbb{C})$ under composition

$$W_{\mathbb{Q}} \times SL(2,\mathbb{C}) \to {}^L H \to {}^L G$$

lies in no proper Levi subgroup of ${}^L G$. We shall examine this question for $PSp(4)$.

3.3. Consider the example of $G = PSp(4)$. As a reductive group over $\mathbb{Q}$, G has only two cuspidal endoscopic groups (up to equivalence) - G itself, and

$$H = H_s \cong PGL(2) \times PGL(2) ,$$

with

$$s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} .$$

Let us look at the formula (3.2.3) in this case. The constant $\iota(G,G)$ equals 1. The group

$$\mathrm{Norm}(sZ_G, {}^L G^0)/{}^L H^0$$

has order 2, the nontrivial element being the coset of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Since

$$Z_H / Z_G \cong \mathbb{Z}/2\mathbb{Z} ,$$

we have

$$\iota(G,H) = \frac{1}{4} .$$

The group H has no proper cuspidal endoscopic group. This means that S_+^H equals I_+^H, and so is given by the formula (3.2.2). Formula (3.2.3) is then equivalent to the assertion that the distribution

$$f \to I_+^G(f) - \frac{1}{4} I_+^H(f_H) \qquad f \in C_c^\infty(G(\mathbb{A})),$$

is stable. Since the distribution

$$f \to I_+^H(f_H)$$

is neither stable nor tempered, the assertion would give interesting information about the discrete spectrum of G.

The one dimensional automorphic representations of H are just

(3.3.1) $$(h_1,h_2) \to \mu_1(\det h_1)\mu_2(\det h_2) , \qquad h_1,h_2 \in PGL(2,\mathbb{A})$$

where μ_1 and μ_2 are Grössencharacters whose images are contained in $\{\pm 1\}$. For any such representation define

$$\psi: W_{\mathbb{Q}} \times SL(2,\mathbb{C}) \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \times W_{\mathbb{Q}} \cong {}^L H$$

by

$$\psi(w,\sigma) = (\mu_1(w')\sigma, \mu_2(w')\sigma, w) ,$$

where w' is the projection of w onto the commutator quotient of $W_{\mathbb{Q}}$, and each $\mu_i(w')$ is identified with a central element in $SL(2,\mathbb{C})$. As we did for real groups, we define a map

$$\phi_\psi: W_{\mathbb{Q}} \to {}^L H$$

as the composition of the map

$$w \to \left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \qquad w \in W_{\mathbb{Q}},$$

with ψ. Then the global L-packet $\prod^H_{\phi_\psi}$ equals $\prod^H_\psi$, and contains exactly one element, the representation (3.3.1). By composing with the natural embedding ${}^LH \subset {}^LG$, we identify each ψ with a mapping of $W_{\mathbb{Q}} \times SL(2,\mathbb{C})$ into ${}^LG$. In this way we obtain parameters in $\Psi(G/\mathbb{Q})$. They are just the ones considered in Examples 2.4.2 and 2.4.3.

The contribution of ψ and H to the right hand side of (3.2.3) equals the product of $\frac{1}{4}$ with the character of the representation (3.3.1) evaluated at f_H. Assume that the Examples 1.4.2 and 1.4.3 for $G(\mathbb{R})$ carry over to each local group $G(\mathbb{Q}_v)$. Then to the local parameters $\psi_v \in \Psi(G/\mathbb{Q}_v)$, obtained from ψ, we have the local packets $\prod_{\psi_v}$. On these packets, the signs ε_{ψ_v} are all 1. If

$$f = \otimes_v f_v, \qquad f_v \in C_c^\infty(G(\mathbb{Q}_v)),$$

the contribution of ψ and H to (3.2.3) is just

$$\begin{aligned} \frac{1}{4} f_H(\psi) &= \frac{1}{4} \prod_v f_{v,H}(\psi_v) \\ &= \frac{1}{4} \prod_v \Big(c_v(s_v) \sum_{\pi_v \in \prod_{\psi_v}} \langle \bar{s}_v, \pi_v \rangle \operatorname{tr} \pi_v(f_v)\Big), \end{aligned}$$

where s_v is the image of s in C_{ψ_v}/Z_G and $\bar{s}_v$ is its projection onto $\mathcal{C}_{\psi_v}$. This becomes

(3.3.2) $$\frac{1}{4} \prod_v \Big(\sum_{\pi_v \in \prod_{\psi_v}} \langle \bar{s}_v, \pi_v \rangle \operatorname{tr} \pi_v(f_v)\Big)$$

if we assume the product formula

$$\prod_v c_v(s_v) = 1 .$$

Suppose that $\mu_1 = \mu_2 = \mu$. The conjecture requires that (3.3.2) should be cancelled by a term in (3.2.2) indexed by (M,σ) with $M \neq G$. The projection of the image of ψ onto ${}^LG^0$ is conjugate to

$$(3.3.3) \qquad \left\{ \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix} : h \in SL(2,\mathbb{C}) \right\} ,$$

a subgroup of

$${}^LM^0 = \left\{ \begin{pmatrix} g & 0 \\ 0 & \alpha(\sigma) \end{pmatrix} \cdot \sigma \in GL(2,\mathbb{C}) \right\} ,$$

where

$$h' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and

$$\alpha(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tg^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

But ${}^LM^0$ is the identity component of the L-group of a Levi subgroup M of G which is isomorphic to $GL(2)$. Set

$$\sigma(m) = \mu(\det(m)) , \qquad m \in GL(2,\mathbb{A}) .$$

Then σ can be regarded as an automorphic representation of M which occurs discretely (modulo the center of $M(\mathbb{A})$). It is the pair (M,σ) whose contribution to (3.2.2) we will compare with (3.3.2).

Let w be a representative in $G(\mathbb{Q})$ of the nontrivial element of the Weyl group $W(\mathfrak{a}_M)$. The representation σ is a lift to $GL(2)$ of an automorphic representation of $PGL(2)$. It is fixed by $\mathrm{ad}(w)$. The contribution of (M,σ) to the formula (3.2.2) for $I_+(f)$ is

(3.3.4) $$\frac{1}{4}\,\mathrm{tr}(T(w)\rho_\sigma(f))\ ,$$

since

$$|W(\mathfrak{a}_M)|^{-1}\ |\det(1-w)_{\mathfrak{a}_M}|^{-1} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\ .$$

We can expect a decomposition

$$T(w) = m(w)\prod_v N_v(w)$$

of $T(w)$ into local normalized intertwining operators. (See [9(b), p. 282].) If ϕ_1 is the three dimensional representation of $M_{\mathfrak{L}}$ obtained by composing ϕ_ψ with the adjoint representation of the group (3.3.3), and $\tilde{\phi}_1$ is its contragradient, the global normalizing factor $m(w)$ equals

$$\lim_{s\to 0}\frac{L(s,\tilde{\phi}_1)}{L(-s,\phi_1)}\ .$$

One checks that it equals 1. Therefore, (3.3.4) equals

$$\frac{1}{4}\prod_v \mathrm{tr}\ (N_v(w)\rho_{\sigma_v}(f_v))\ ,$$

where σ_v is the character $\mu_v(\det(\cdot))$ on $GL(2,\mathfrak{Q}_v)$, with μ_v the local component of the Grössencharacter μ. With a resolution to Problem 1.4.4, or rather its analogue for each place v, the expression would become

$$\frac{1}{4}\prod_v\Big(\sum_{\pi_v\in\Pi_{\psi_v}}\langle\bar{s}_v,\pi_v\rangle\mathrm{tr}\ \pi_v(f_v)\Big)\ .$$

This is just (3.3.2).

Thus, when $\mu_1 = \mu_2 = \mu$, so that ψ factors through a Levi subgroup, the contribution of ψ and H to (3.2.3) would be completely cancelled by a term in (3.2.2) with $M \neq G$. This suggests that such ψ contribute nothing to the discrete spectrum of $G(\mathbb{A})$, as predicted by the conjecture.

3.4. In order for the two terms above to cancel, it was essential that

$$\iota(G,H) = |W(\mathfrak{a}_M)|^{-1} |\det(1-w)_{\mathfrak{a}_M}|^{-1} ,$$

the common value, we recall, being $\frac{1}{4}$. This fact may be interpreted as a combinatorial property of the complex group

$$C_\psi = O(2,\mathbb{C}) .$$

The generalization of this property will be a key to affecting similar cancellations for arbitrary groups. We shall describe it.

Let C be the set of complex points of a complex reductive algebraic group. We do not assume that C is connected. Let C^0 be the identity component of C. Let T^0 be a Cartan subgroup of C^0, and let W be the normalizer of T^0 in C, modulo T^0. Then W is an extension of

$$W^0 = W \cap C^0 ,$$

the Weyl group of (C^0,T^0). It acts on T^0 and on its Lie algebra. Let W_{reg} be the set of elements in W for which 1 is not an eigenvalue. If w is any element in W, set

$$\varepsilon(w) = (-1)^{n(w)} ,$$

where $n(w)$ equals the number of positive roots of (C^0,T^0) which are mapped by w to negative roots. ($\varepsilon(w)$ is independent of how the positive roots are chosen.) For each connected component x of C we define

$$i(x) = |W^0|^{-1} \sum_{w \in W_{reg}(x)} \varepsilon(w) |\det(1-w)|^{-1} ,$$

where $W_{reg}(x)$ is the set of elements in W_{reg} induced from points in x. The number $i(x)$ is a sort of scalar analogue of the invariant distribution (3.2.2).

For each component x of C, let $\mathrm{Orb}(C^0,x)$ be the set of C^0-orbits of elements in x for which the adjoint map (as a linear operator on the Lie algebra of C^0) is semisimple. If s belongs to any of the orbits, the group

$$C_s = \mathrm{Cent}(s,C^0)$$

satisfies the same hypothesis as C. Its conjugacy class in C^0 depends only on the orbit of s. The number

$$|C_s/C_s^0|^{-1}$$

of connected components in C_s also depends only on the orbit of s. It is possible to define uniquely a number $\sigma(C)$, for every group C, which depends only on C^0, and vanishes unless the center of C^0 is finite, such that

$$i(C^0) = \sum_{s\in \mathrm{Orb}(C^0,C^0)} |C_s/C_s^0|^{-1}\sigma(C_s) \tag{3.4.1}$$

for every group C. Indeed, there are only finitely many orbits s in $\mathrm{Orb}(C^0,C^0)$ such that the center of C_s is finite, so we can define $\sigma(C)$ inductively by this last equation. We see inductively that it depends only on C^0. The numbers $\sigma(C)$ are scalar analogues of the stable distribution defined by (3.2.3).

<u>Theorem 3.4.2</u>: With the possible exclusion of the case that C^0 has exceptional simple factors, we have

$$i(x) = \sum_{s\in \mathrm{Orb}(C^0,x)} |C_s/C_s^0|^{-1}\sigma(C_s), \tag{3.4.3}$$

for every component x of C.

The details will appear in [1(c)]. (I have not yet had a chance to look at the exceptional groups.)

Equations (1.3.6), (3.1.2) and (3.4.1) are all in the same spirit. They each provide an inductive definition for a set of objects (stable distributions, for example) in terms of given objects (such as invariant distributions). The inductive definition in each case is by a sum over indices which are closely related to endoscopic groups. Equations (1.3.6) and (3.1.2) should have twisted analogues. These should be true identities, involving the objects defined by the original equations. The twisted analogue of (3.4.1) we have just encountered. It is the formula (3.4.3).

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Note added in proof: The sign function ε_ψ in the local Conjecture 1.3.3 and the sign character ξ_ψ in the global Conjecture 2.1.1 should both have simple formulas.

Suppose that

$$\psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^LG$$

is given as in Conjecture 1.3.3. Then

$$s_\psi = \psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

belongs to the centralizer C_ψ. Let $\bar{s}_\psi$ be the image of s_ψ in $\mathcal{C}_\psi$. Then ε_ψ should be given in terms of the pairing on $\mathcal{C}_\psi \times \Pi_\psi$ by

$$\varepsilon_\psi(\pi) = \langle \bar{s}_\psi, \pi \rangle, \qquad \pi \in \Pi_\psi.$$

In particular, if the unipotent element

$$\psi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

in ${}^LG^0$ is even, the function ε_ψ will be identically 1.

Suppose that F is global and

$$\psi : G_{\Pi_{\text{temp}}(F)} \times SL(2,\mathbb{C}) \to {}^LG$$

is given as in Conjecture 2.1.1. Assume that C_ψ is finite. Let $\mathfrak{g}$ be the Lie algebra of ${}^LG^0$, and define a finite dimensional representation

$$r_\psi : C_\psi \times G_{\Pi_{\text{temp}}(F)} \times SL(2,\mathbb{C}) \to GL(\mathfrak{g})$$

by

$$r_\psi(c,w,g) = \mathrm{Ad}(c \cdot \psi(w,g)),$$

for $c \in C_\psi$, $w \in G_{\Pi_{\text{temp}}(F)}$ and $g \in SL(2,\mathbb{C})$. Then there is a decomposition

$$r_\psi = \oplus_{i\in I_\psi} (\xi_i \otimes \phi_i \otimes \rho_i)$$

where ξ_i, ϕ_i and ρ_i are irreducible (finite-dimensional) representations of C_ψ, $G_{\Pi_{\text{temp}}(F)}$ and $SL(2,\mathbb{C})$ respectively. Suppose that for a given i, the representation ϕ_i is equivalent to its contragredient. Then from the anticipated functional equation of the L-function $L(s,\phi_i)$, we see that

$$\varepsilon(\tfrac{1}{2},\phi_i) = \pm 1.$$

Let I_ψ^- be the set of such indices i such that $\varepsilon(\frac{1}{2},\phi_i)$ actually equals -1, and such that in addition, the dimension of ρ_i is even. Then the sign character should be given by

$$\xi_\psi(c) = \prod_{i\in I_\psi^-} \det(\xi_i(c)), \qquad c \in C_\psi.$$

Such a formula (assuming it is true) is rather intruiging. It ties the values of ε-factors at $\frac{1}{2}$ in an essential way to multiplicities of cusp forms, and it also suggests that the adjoint representation of the L-group might play some role in questions of L-indistinguishability.