Motives and Automorphic Representations

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Contents

References: [A1] J. Arthur, A note on the Automorphic Langlands Group, Bull. Canad. Math Soc. 45 (2002), 466-482.

[A2] ———, The work of Robert Langlands, arXiv, to appear in H. Holden and R. Piene, The Abel Laureates 2018-2022, Springer, 2024

[A3] ———, Motives and automorphic representations, in preparation

(2)

1. Motives (pure), Grothendieck (c 1965)

2 simultaneous roles:

(i) Fundamental (but hidden) building blocks of smooth, projective, alg. varieties

(ii) Universal cohomology theory, through which all other cohom. theories in algebraic geom. factor. (Betti, deRham, l-adic, crystalline, Hodge theory, etc.)

- F, Q : fields of char 0, $F \subset \mathbb{C}$.
  (Usually F is number field + $Q = \mathbb{Q}$)

- Expect : a semisimple, Q-linear category $\mathrm{Mot}_{F,Q}$,
  pure motives over F with coeffs in Q.

- Would come with 2 functors

$$\underbrace{(\mathrm{SProj})_F}_{\text{smooth proj. varieties over } F} \xrightarrow{M_F} \mathrm{Mot}_{F,Q}$$

and

$$\mathrm{Mot}_{F,Q} \xrightarrow{R_B} \underbrace{\mathrm{Vect}_Q}_{\text{vector spaces over } Q}$$

whose composition

$$H_B = R_B \circ M_F,$$

$$(\mathrm{SProj})_F \longrightarrow \underbrace{\mathrm{Mot}_{F,Q}}_{\text{building blocks}} \xrightarrow[\text{universal cohomology}]{} (\mathrm{Vect})_Q$$

is just Betti (singular) cohom. of complex manifolds with Q-coefficients.

(4)

- $M_{F,\mathbb{Q}}$ would be a Tannakian category with fibre functor H_B; this means that $Mot_{F,\mathbb{Q}}$ is (anti) isomorphic to the category $Rep_{\mathbb{Q}}(\mathcal{G}_{F,\mathbb{Q}})$ of finite dim. representations of a reductive, proalgebraic gp $\mathcal{G}_{F,\mathbb{Q}}$ defined over $\mathbb{Q}$
- Assume now F is a number field + $\mathbb{Q} = \mathbb{Q}$.

Then $\mathcal{G}_F = \mathcal{G}_{F,\mathbb{Q}}$ would be a group over $\mathbb{Q}$, with a canonical mapping

$$\mathcal{G}_F \longrightarrow \Gamma_F, \qquad \Gamma_F = Gal(\bar{F}/F)$$

<u>What is it</u>!

(5)

2. Automorphic Representations, Langlands (1967)

- Letter to Weil (1967),
- "Problems in the theory of automorphic forms (1970"

G - reductive alg. gp over F

Recall: $G(F)$ (discrete) $\subset G(\mathbb{A}_F)$ (locally compact)

- Informal def: An automorphic rep. $\pi \in \Pi_{aut}(G)$ is an irreducible representation that "occurs in" the decomp. of $L^2(G(F) \backslash G(\mathbb{A}_F))$
- $\pi = \widetilde{\bigotimes_v} \pi_v$ - (restricted) direct product of irred reps $\pi_v \in \Pi(G_v)$ of the local completions $G_v = G(F_v)$, almost all of which,

$\{\pi_v : v \notin S\}$, ← finite set of valuations including archimedean places S_∞

are unramified.

They should be governed / classified by certain loc. c^{pt} groups attached to F + its local completions:

$$\begin{tikzcd} L_{F_v} \arrow[r, two heads] \arrow[d, hook] & W_{F_v} \arrow[r] \arrow[d, hook] & \Gamma_{F_v} \arrow[d, hook] \\ L_F \arrow[r, two heads] & W_F \arrow[r] & \Gamma_F \end{tikzcd},$$

embeddings defined up to conjugacy

"Langlands group" — Weil group — Galois group

for the local Langlands groups $L_{F_v} = \begin{cases} W_{F_v}, & v \in S_\infty, \\ W_{F_v} \times SU(2), & v \notin S_\infty, \end{cases}$

-well understood-, and the hypothetical global Langlands gp L_F (automorphic Galois group) -not well understood-, a precursor to the motivic Galois gp.

Those locally compact gp^s are ingredients for a conjectured classification of irred. rep^s of $G_v = G(F_v)$, and aut. rep^s

$$\pi = \tilde{\bigotimes_v} \pi_v, \qquad \pi_v \in \Pi_{irred}(G_v),$$

of $G(\mathbb{A})$ - very deep.

(7)

3. Conjectural candidate for aut. Galois gp L_F [AL], [A2]

2 ingredients

(i) An indexing set $\mathcal{C}_F = \{(G, c)\}$, where G/F is a quasisplit, simple, simply connected gp over F, and

$$c = \{c_v : v \notin S\}$$

is a concrete datum attached to a primitive* aut. rep. π of $G(\mathbb{A})$ ~

(the set of conj. classes $\{c_v(\pi) = c(\pi_v) : v \notin S\}$ in the local Langlands "L-groups" ${}^L G_v = \hat{G} \rtimes \Gamma_{F_v}$ that classify unramified reps $\{\pi_v : v \notin S\}$.)

(ii) For each $c \sim (G, c)$ in $\mathcal{C}_F$, an extension

$$1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_F \longrightarrow 1$$

of W_F (the global Weil group) by a compact simply connected group K_c (a compact real form of the simply connected cover $\hat{G}_{sc}$ of the complex dual group $\hat{G}$.

* primitive: not a proper functorial image $\Longleftrightarrow$ $\mathrm{ord}_{s=1}(L(s, \pi, r))$ is minimal $\forall r$.

Given (i) + (ii), we define the automorphic Galois gp

$$L_F = \prod_{c \in \mathcal{C}_F} (L_c \to W_F) - \text{a fibre product over } W_F$$

– a locally compact gp, with local embeddings

$$\begin{tikzcd} L_{F_v} \arrow[r, mapsto] \arrow[d, hook] & W_{F_v} \arrow[d, hook] \\ L_F \arrow[r, twoheadrightarrow] & W_F \end{tikzcd}$$

Note: Both (i) + (ii) depend on Langlands conjectural

Principle of Functoriality

(for the values of global L-functions $L^S(s, c, r)$ near $s=1$, in order to define primitive aut. rep. π.)

Proposition [A3] An explicit identity

$$L_{\mathbb{Q}}/L_F \cong W_{\mathbb{Q}}/L_F \cong \Gamma_{\mathbb{Q}}/\Gamma_F \cong \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}),$$

generalizing Tate (Corvallis, p. 3), + based on Clozel-Rajan, Solvable base change for $GL(n)$

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Conjecture 1: There is a bijection

$\{$ Irred. rep^s $r: L_F \to GL(N, \mathbb{C})\} \xrightarrow{\sim}$

$\{$ cuspidal aut. rep^s of $GL(N, \mathbb{A})\}$,

which is compatible with localizations F_v,

Conjecture 2: A more general mapping

$\{\phi : L_F \to {}^LG\} \xrightarrow{\sim} \{\Pi_\phi \subset \Pi_{aut, temp}(G)\}$

for any G/F quasisplit, from bounded, global Langlands parameters to disjoint global L-packets, whose union ~~is~~ the set $\Pi_{aut, temp}(G)$ of tempered, aut. rep^s of $G(\mathbb{A})$

4. Conjectural candidate for motivic Galois gp $\mathcal{G}_F$

[A1], [A2]

The const^n of L_F leads to a complex, pro-alg. gp $\mathcal{G}_F$, depending on the embedding $F \subset \mathbb{C}$, with maps

$$\begin{tikzcd}
L_F \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d] & \Gamma_F \arrow[d, equal] & \text{loc. compact gp}^s \arrow[l] \\
\mathcal{G}_F \arrow[r] & \mathcal{T}_F \arrow[r] & \Gamma_F & \text{complex, proalg. gp}^s \arrow[l]
\end{tikzcd}$$

complex, proalg. gp^s whose reps $M: \mathcal{G}_F \to GL(N, F)$ would parametrize motives

fibre product of complexifications $\mathcal{G}_c$ of L_c, for $c \in \mathcal{C}_{F,Hod} \subset \mathcal{C}_F$ of Hodge type – i.e. such that $\forall\ r: L_c \to GL(n, \mathbb{C})$, & $\forall\ v \in S_\infty$, the rest^n of r to the sub^{gp} $\mathbb{C}^* \subset W_{F_v} = L_{F_v} \hookrightarrow L_c$ is a Hodge structure

Langlands' Taniyama group (Corvallis conf. proceedings, 1979) – the "algebraic hull" of the "motivic part" of global Weil gp W_F

Thus, we get

$$\mathcal{G}_F = \widetilde{\prod_{c \in \mathcal{C}_{F,\mathrm{Hod}}}} \left(\mathcal{G}_c \longrightarrow \mathcal{T}_F \right)$$

a fibre product over $\mathcal{T}_F$; a proalg. reductive gp. over $\mathbb{C}$ that comes with local embeddings

$$\begin{array}{ccc} L_{F_v} & \longrightarrow & W_{F_v} \\ \downarrow & & \downarrow \\ \mathcal{G}_F & \longrightarrow & \mathcal{T}_F \end{array}$$

This would be a general form of the Shimura-Taniyama-Weil conjecture (Langlands Principle of Reciprocity) whose proof we can hope is tied up in the (future) proof of Langlands Principle of Functoriality

(12)

Properties of $\mathcal{G}_F$ [A3]

(i) $\mathbb{Q}$-structure on $\mathcal{G}_F$

For each (G, c) in $\mathcal{G}_{F,\mathrm{Hod}}$ that parametrizes a factor $\mathcal{G}_c$ of $\mathcal{G}_F$, an explicit $\mathbb{Q}$-structure (conjectural) on the complex group $\mathcal{G}_c$.

(ii) Cohomological realizations for $\mathcal{G}_F$

An explicit (conjectural) description of the cohomology groups for different theories attached to any motive M ~ i.e. any fin. dim rep. over $\mathbb{Q}$

$$M : \mathcal{G}_F \longrightarrow GL(V), \qquad V = V_{\mathbb{Q}}.$$

iii) Motivic periods

An explicit (conjectural) comparison of Betti cohomology $H_B(M) \overset{\mathrm{def}}{=} V(\mathbb{Q})$ with de Rham cohomology $H_{DR}(M)$, a finite dim. vect sp. over F. This is an explicit iso:

$$\omega_M : H_{DR}(M) \otimes_F \mathbb{C} \xrightarrow{\sim} H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}.$$

If $F = \mathbb{Q}$, ω_M can be expressed as a complex, square matrix, whose entries span the periods of M

5. Mixed motives

We should also have the mixed motivic Galois group $\mathcal{G}_F^+ = \mathcal{U}_F \rtimes \mathcal{G}_F$, with

$$\mathcal{G}_F^+ \longrightarrow \mathcal{G}_F \longrightarrow \mathcal{T}_F \longrightarrow \Gamma_F,$$

which plays role of $\mathcal{G}_F$ for singular/open varieties V/F.

Problem: Find an explicit (conjectural) construction for its unip. radical $\mathcal{U}_F$, as a prounipotent alg. $\mathfrak{g}^p/\mathbb{Q}$, with $\mathbb{Q}$-action of $\mathcal{G}_F$.

Let $\mathfrak{n}_F$ be Lie alg. of $\mathcal{U}_F$, with commutator quotient $\mathfrak{n}_F^{ab} = \mathfrak{n}_F/\mathfrak{n}_F^c$ - a proj. limit of finite dim. vector spaces over $\mathbb{Q}$, with linear action of

$$\mathcal{G}_F = \widehat{\prod_{c \in \mathcal{B}_{F,Hod}}} (\mathcal{G}_c \longrightarrow \mathcal{T}_F)$$

Question: What is its the decomp. of $\mathfrak{n}_F^{ab}$ into irred. rep^s of $\mathcal{G}_F$ over $\mathbb{Q}$.

Answer: Conjectures of Deligne, Beilinson, Bloch, ...

(14)

<u>Project</u>: Express this answer <u>explicitly</u> in terms of irred. rep^s of the simp. conn. complex gp^s $\hat{G}_{sc}$ in the factors $\mathcal{G}_c$ of $\mathcal{G}_F$.

Could then describe $\mathcal{U}_F = \exp(\mathfrak{n}_F)$ explicitly as the <u>free</u> <u>prounipotent</u> <u>group</u>, <u>with</u> <u>action</u> <u>of</u> <u>$\mathcal{G}_F$</u> <u>over</u> <u>$\mathbb{Q}$</u>, <u>based</u> <u>on</u> <u>the</u> <u>$\mathcal{G}_F$-module</u> <u>$\mathfrak{n}_F^{ab}$</u> (Lubotzky - Magid). This is implied by the conj. of Beilinson

$$\mathrm{Ext}^i_{\mathcal{MM}_F}(\mathbb{Q}(0), M) = 0, \qquad i \geq 2,$$

for any pure motive $M \in \mathcal{M}_F$.

<u>We</u> <u>would</u> <u>then</u> <u>have</u> <u>an</u> <u>explicit</u> (conjectural) <u>construction</u> <u>of</u> <u>the</u> <u>mixed</u> <u>motivic</u> <u>Galois</u> <u>group</u> $\tilde{\mathcal{G}}_F$.

<u>Note</u>: Conjectures of Beilinson contain more information than this; towards <u>explicit</u> <u>formulas</u> for the <u>periods</u> of $\mathcal{MM}_F$.

6. Exponential motives

An important generalization of motives:

i) Gives periods (such as e, γ) not attached to ordinary motives.

ii) Is needed for fund. physics (Kontsevich-Soibelman), and

iii) Plays role of diff. eq^{ns} with irregular sing. points in Riemann-Hilb. corresp. (Deligne, Katz, Bloch, Esnault; Fresan-Jossen)

They give (conjectural) exponential motivic Galois group $\widehat{\mathcal{G}}_F$ + its mixed ext^n $\widehat{\mathcal{G}}_F^+ = \widehat{\pi}_F \rtimes \widehat{\mathcal{G}}_F$, with

ext^n of $\mathfrak{T}_F$ defined by Greg Anderson

$$
\begin{tikzcd}
\widehat{\mathcal{G}}_F^+ \arrow[r] \arrow[d] & \widehat{\mathcal{G}}_F \arrow[r] \arrow[d] & \widehat{\mathfrak{T}}_F \arrow[r] \arrow[d] & \Gamma_F \arrow[d, equal] \\
\mathcal{G}_F^+ \arrow[r] & \mathcal{G}_F \arrow[r] & \mathfrak{T}_F \arrow[r] & \Gamma_F .
\end{tikzcd}
$$

<u>Basic idea</u>: Given $F \subset \mathbb{C}$, replace varieties X/F with pairs

$$(X, f), \qquad f : X \longrightarrow F \text{ regular,}$$

<u>varieties</u> <u>with</u> <u>potential</u>. Define

$$H_*^{rd}(X,f) = \lim_{r \to \infty} H_*(X(\mathbb{C}), f^{-1}(S_r); \mathbb{Q}),$$

(relative Betti homology), with $S_r = \{z \in \mathbb{C} : Re(z) \geq r\}$, and

$$H^*_{rd}(X,f) = \mathrm{Hom}_{\mathbb{Q}}(H_*^{rd}(X,f), \mathbb{Q}).$$

(<u>rapid</u> <u>decay</u> homology + <u>cohomology</u>). It can be shown that there is de Rham cohomology $H^*_{DR}(X,f)$ of X + a canonical isom

$$H^*_{DR}(X,f) \otimes_F \mathbb{C} \xrightarrow{\sim} H^*_{rd}(X,f) \otimes_{\mathbb{Q}} \mathbb{C}$$

~ <u>exponential</u> <u>period</u> <u>mapping</u>

Problem: Find explicit (conjectural) construction for $\hat{\mathcal{G}}_F$ like that of $\mathcal{G}_F$. For this, would need further generalization of S-T-W conjecture – i.e. an exponential automorphic Galois group $\hat{L}_F$, with

$$\begin{tikzcd} \hat{L}_F \arrow[r] \arrow[d] & \tilde{W}_F \arrow[r] \arrow[d] & \Gamma_F \arrow[d, equal] \\ L_F \arrow[r] & W_F \arrow[r] & \Gamma_F \end{tikzcd}$$

What could it possibly be?

Conjecture: $\hat{L}_F$ comes from automorphic rep^s of topological covering groups

$$\hat{G}(\mathbb{A}) \longrightarrow G(\mathbb{A}),$$

attached to Brylinski–Deligne ext^{ns}