Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups of Real Rank One

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SUMMARY. Let G be a real semisimple Lie group. Harish-Chandra has defined the Schwartz space, $\mathscr{C}(G)$, on G. A tempered distribution on G is a continuous linear functional on $\mathscr{C}(G)$.

If the real rank of G equals one, Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [**3(k)**, §24]. We restrict the Fourier transform map to $\mathscr{C}(G)$, and we compute the image of the space $\mathscr{C}(G)$ [Theorem 3]. This permits us to develop the theory of harmonic analysis for tempered distributions on G [Theorem 5].

CONTENTS

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§1. **Introduction.** Let G be a real semisimple Lie group. The Fourier transform map, $\mathscr{F}$, can be regarded as an isometry form $L^2(G)$ onto $L^2(\hat{G})$. $L^2(\hat{G})$ is a Hilbert space defined with the help of the discrete series, $\mathscr{E}_d$, and the various continuous series, $\mathscr{E}_c$, of irreducible unitary representations of G. $L^2(\hat{G})$ consists of certain functions whose domain is $\mathscr{E}_d \cup \mathscr{E}_c$, and whose range is the space of Hilbert-Schmidt operators on the Hilbert spaces on which the representations in $\mathscr{E}_d \cup \mathscr{E}_c$ act.

In [**3(l)**] Harish-Chandra introduces the Schwartz space, $\mathscr{C}(G)$, of functions on G. It is analogous to the space, $\mathscr{S}(\mathbf{R})$, of rapidly decreasing functions on the real line. $\mathscr{C}(G)$ is a Fréchet space. It is dense in $L^2(G)$, and its injection into $L^2(G)$ is continuous. It is of interest to ask about the image of $\mathscr{C}(G)$ in $L^2(\hat{G})$ under $\mathscr{F}$. There is a candidate, $\mathscr{C}(\hat{G})$, for this image space. $\mathscr{C}(\hat{G})$ is a Fréchet space defined by a natural family of seminorms on $L^2(\hat{G})$.

A tempered distribution on G is a continuous linear functional on $\mathscr{C}(G)$. If we can prove that the Fourier transform gives a topological isomorphism from $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$, we could define the Fourier transform of a tempered distribution as a continuous linear functional on $\mathscr{C}(\hat{G})$. This would include as a special case the theory of Fourier transforms on $L^2(G)$.

We confine ourselves to the case in which the real rank of G equals one. In this case Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [**3(k)**, §24]. Our main result is Theorem 3, which asserts the bijectivity between $\mathscr{C}(G)$ and $\mathscr{C}(\hat{G})$ of the Fourier transform, $\mathscr{F}$.

The most difficult part of this theorem is to prove surjectivity. We have to show that the inverse Fourier transform of an element in $\mathscr{C}(\hat{G})$, which is *a priori* in $L^2(G)$, is actually in $\mathscr{C}(G)$. We use some estimates which Harish-Chandra develops from the study of a differential equation on G [**3(l)**, §27]. In §9 we review his work and show that his estimates are actually uniform, in a sense which will become clear. In §10 we use these estimates to prove that $\mathscr{F}(\mathscr{C}(G))$ contains $\mathscr{C}_0(\hat{G})$, a subspace of $\mathscr{C}(\hat{G})$ associated with the discrete series.

To prove that $\mathscr{F}(\mathscr{C}(G))$ contains $\mathscr{C}_1(\hat{G})$, the subspace of $\mathscr{C}(\hat{G})$ associated with the continuous series, requires more work. It is necessary to derive a formula (Lemma 41) for the norms of certain linear transformations, $c^+(\Lambda)$ and $c^-(\Lambda)$, which arise in §12. This we do in §13 by studying a second-order symmetric differential operator on $\mathfrak{a}_\mathfrak{p}$, a one-dimensional subspace of the Lie algebra of G. As a biproduct of this formula we obtain in §14 a condition for irreducibility of certain representations in the continuous series.

For convenience we work with generalized spherical functions. We develop the pertinent information in §5 and then use it in §6 to prove the injectivity of the Fourier transform.

In §16 we define the Fourier transform of a tempered distribution on G. Theorem 6 proves that any continuous linear functional on $\mathscr{C}(\hat{G})$ is a certain sum of tempered distributions on the real line.

It seems likely that some of our methods can be used for proving the analogue of Theorem 3 for arbitrary G. The injectivity of the Fourier transform should carry over quite easily. Harish-Chandra's estimates are proved in [**3(l)**, §27] for arbitrary G. That these estimates are uniform can also be shown, although the proof of this is somewhat more complicated than in the real rank 1 case. Our proof of Lemma 27 does not carry over in general. However, it gives a good start toward a general proof.

The general Plancherel formula will be complicated by the existence of more than one continuous series of representations. However, in each continuous series linear transformations $c(\Lambda)$ can be defined. The formulae in Lemma 41 can probably be proved, although perhaps not by our methods. In general, Lemma 44 would be proved by induction on the real rank of G. Harish-Chandra does this for ordinary spherical functions in [**3(h)**, Theorem 3].

2. Preliminaries. Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a fixed Cartan decomposition with Cartan involution θ. Let $\mathfrak{a}_\mathfrak{p}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. The dimension of $\mathfrak{a}_\mathfrak{p}$ is called the real rank of G. We shall assume that $\dim \mathfrak{a}_\mathfrak{p} = 1$.

Let $\mathfrak{a}_\mathfrak{k}$ be a subspace of $\mathfrak{k}$ such that

$$\mathfrak{a} = \mathfrak{a}\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$$

is a Cartan subalgebra of $\mathfrak{g}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. We assume that G has finite center. This implies that K is compact.

We can make further technical assumptions on G without losing generality. In order to do this we state some definitions of Harish-Chandra.

If L is a connected reductive Lie group over the reals, $\mathbf{R}$, with Lie algebra $\mathfrak{l}$, let

$$j : \mathfrak{l} \subset \mathfrak{l}_c$$

be inclusion into the complexification of $\mathfrak{l}$. (From now on, if $\mathfrak{h}$ is any real Lie algebra we write $\mathfrak{h}_c$ for its complexification.) Then if L_c is a complex analytic group with Lie algebra $\mathfrak{l}_c$, L_c is called a complexification of L if j extends to a homomorphism of L into L_c. Let $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{c}$, where $\mathfrak{l}_1$ is semisimple and $\mathfrak{c}$ is abelian. Let $\mathfrak{l}_{1c}$ and $\mathfrak{c}_c$ be the respective complexifications of $\mathfrak{l}_1$ and $\mathfrak{c}$. Let L_1, $C(L_{1c}, C_c)$ be the analytic subgroups of $L(L_c)$ corresponding to $\mathfrak{l}_1$, $\mathfrak{c}(\mathfrak{l}_{1c}, \mathfrak{c}_c)$ respectively. We call L_c quasi-simply connected (q.s.c.) if $L_{1c} \cap C_c = \{1\}$ and if L_{1c} is simply connected. We say that L is q.s.c. if it has a q.s.c. complexification.

Fix a complexification $j: L \to L_c$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Let A and A_c be the Cartan subgroups of L and L_c corresponding to $\mathfrak{h}$ and $\mathfrak{h}_c$ (that is, the centralizers of $\mathfrak{h}$ and $\mathfrak{h}_c$ in G and G_c respectively). Clearly $j(A) \subseteq A_c$. It is known that A_c is connected [**3(j)**, corollary to Lemma 27]. If λ is a linear functional on $\mathfrak{h}_c$, there exists at most one complex analytic homomorphism

$$\xi_\lambda : A_c \to \mathbf{C}$$

such that for every H in $\mathfrak{h}_{\mathbf{c}}$

$$\xi_\lambda(\exp H) = e^{\lambda(H)}.$$

We also write ξ_λ for the homomorphism

$$\xi_\lambda \circ j \colon A \to \mathbf{C}.$$

ξ_λ can be seen to be independent of the complexification $L_{\mathbf{c}}$ used, provided that ξ_λ is defined on that complexification.

Clearly ξ_α exists for any root α of $(\mathfrak{l}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$. If $P_{\mathfrak{h}}$ is the set of positive roots relative to some ordering, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P_{\mathfrak{h}}} \alpha.$$

It is easy to see that the question of the existence of ξ_ρ is independent of the ordering of the roots of $(l_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$ and of the choice of Cartan subalgebra $\mathfrak{h}$. If ξ_ρ exists we call $L_{\mathbf{c}}$ acceptable. We say that L is acceptable if it has an acceptable complexification.

If $L_{\mathbf{c}}$ is q.s.c., it is known that it is acceptable [**3(j)**, Lemma 29]. If $L_1 \cap C$ is finite, it is clear that L has a finite, and hence acceptable, cover.

Suppose L is a compact, connected acceptable Lie group with Lie algebra $\mathfrak{l}$. Let $\mathfrak{h}$, $P_{\mathfrak{h}}$, A, and ρ be defined as above. For each α define an element H_α in $\mathfrak{h}_{\mathbf{c}}$ by

$$B(H_\alpha, H) = \alpha(H)$$

for all H in $\mathfrak{h}_{\mathbf{c}}$, where B is the Killing form of $\mathfrak{h}_{\mathbf{c}}$ restricted to $\mathfrak{h}_{\mathbf{c}}$. Put

$$\tilde{\omega} = \prod_{\alpha \in P} H_\alpha.$$

$\tilde{\omega}$ is in $S(\mathfrak{h}_{\mathbf{c}})$, the symmetric algebra on $\mathfrak{h}_{\mathbf{c}}$. Let Π be the lattice of linear functionals

$$\lambda \colon (-1)^{1/2}\mathfrak{h} \to \mathbf{R}$$

for which ξ_λ exists. Let $\Pi' = \{\lambda \in \Pi \colon \tilde{\omega}(\lambda) \neq 0\}$. If W is the Weyl group of $(l_{\mathbf{c}}/\mathfrak{h}_{\mathbf{c}})$, W acts on $(-1)^{1/2}\mathfrak{h}$. Then W acts on Π as follows:

$$s\mu(H) = \mu(s^{-1}H)$$

for μ in Π, s in W, and H in $(-1)^{1/2}\mathfrak{h}$. For s in W, put $\varepsilon(s) = (-1)^{n(s)}$, where $n(s)$ is the number of positive roots that are mapped by s into negative roots. For h a regular element of A, put

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha \in P_{\mathfrak{h}}} (1 - \xi_\alpha(h^{-1})).$$

LEMMA 1. *There is a map $\mu \to \sigma(\mu)$ from Π' onto the set of unitary equivalence classes of irreducible representations of L. $\sigma(\mu_1) = \sigma(\mu_2)$ if and only if $\mu_1 = s\mu_2$ for some s in W. Furthermore, if h is a regular element of A,*

$$\operatorname{tr} \sigma(\mu)(h) = (\operatorname{sign} \tilde{\omega}(\mu)) \cdot \Delta(h)^{-1} \cdot \left(\sum_{s \in w} \varepsilon(s) \xi_{s\mu}(h) \right).$$

Also there exists a constant c_L, independent of μ, such that

$$\dim \sigma(\mu) = c_L |\tilde{\omega}(\mu)|.$$

Finally, if μ is in Π', and $B(\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$, then $\mu - \rho$ is the highest weight of the representation of the Lie algebra $\mathfrak{l}_c$ corresponding to $\sigma(\lambda)$.

PROOF. Since L has a finite q.s.c. cover, we will assume without loss of generality that L is q.s.c. We can assume further that L is semisimple. Then L is simply connected, so Π is precisely the lattice of weights of $\mathfrak{h}$ [**3(j)**, Lemma 29]. If μ' is a dominant integral function (in the terminology of [**5**, p. 215]), and if $\mu = \mu' + \rho$, then $B(\mu, \alpha) > 0$ for any α in $P_{\mathfrak{h}}$ so μ is in Π'. Conversely, if μ is in Π', there exists a unique s such that $B(s\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$. Then $\mu' = \mu - \rho$ is a dominant integral function on $\mathfrak{h}$. This demonstrates the relation between μ and the highest weight of $\sigma(\mu)$. The correspondence between representations and dominant integral functions is well known (see [**5**, Chapter VII]).

The other two statements of the lemma follow from the Weyl character formula [**5**, p. 255] and the Weyl dimension formula [**5**, p. 257]. □

Now let us return to our group G. By going to a finite cover we can assume that G is q.s.c. and hence acceptable. Thus, if $j: \mathfrak{g} \subset \mathfrak{g}_c$ and G_c is a simply connected analytic group with Lie algebra $\mathfrak{g}_c$ then j extends to a homomorphism

$$j: G \to G_c.$$

Now K is reductive. Therefore, by going to a further finite cover of G, we may also assume that K is acceptable.

If we understand the harmonic analysis of a finite cover, $\tilde{G}$, of G then we understand the theory for G. We merely throw out those unitary representations of $\tilde{G}$ which are nontrivial on the kernel of the covering projection. Therefore, the above two assumptions can be made with no loss of generality.

There are two possibilities for G.

Case I. There exists a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{b}$ is contained in $\mathfrak{k}$. We can assume that $\mathfrak{b}$ has been chosen so that it contains $\mathfrak{a}_{\mathfrak{k}}$. Then it is known that $\{\mathfrak{b}, \mathfrak{a}\}$ is a set of representatives of conjugacy classes of Cartan subalgebras of $\mathfrak{g}$.

Case II. Such a $\mathfrak{b}$ does not exist. Then there is only one conjugacy class of Cartan subalgebras and it is represented by $\mathfrak{a}$.

We shall try as far as possible to deal with these two cases together. Whenever we speak of $\mathfrak{b}$, we shall be implicitly referring to Case I. However, any mention of $\mathfrak{a}$, unless otherwise stated, will refer to either case.

Let B be the Cartan subgroup of G corresponding to $\mathfrak{b}$. Since it is a maximal abelian subgroup in the compact connected Lie group K, it is connected [**4**, Corollary 2.7, p. 247].

Let A be the Cartan subgroup of G corresponding to $\mathfrak{a}$. Then

$$A = A_I A_{\mathfrak{p}}$$

where $A_{\mathfrak{p}} = \exp \mathfrak{a}_{\mathfrak{p}}$, and A_I is contained in K. In Case II, A_I is a Cartan subgroup of K and is connected. Otherwise, A_I may not be connected. In any case, let $\mathfrak{m}$ and M be the centralizers of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$ and K, respectively. Then M is compact with a finite number of connected components.

Fix compatible orders on the real dual spaces of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}} + (-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$. Let P be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$ relative to this order. Let P_+ be the set of roots in P which do not vanish on $\mathfrak{a}_{\mathfrak{p}}$ and let P_M equal $P - P_+$. $\mathfrak{a}_{\mathfrak{k}}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{m}$ and we can regard P_M as the set of positive roots of $(\mathfrak{m}, \mathfrak{a}_{\mathfrak{k}})$.

Let M^0 and A_I^0 be the connected components of M and A_I. Let W and W_1 be the Weyl groups of $(\mathfrak{g}/\mathfrak{a})$ and $(\mathfrak{m}/\mathfrak{a}_{\mathfrak{k}})$, respectively. Now in any connected component of M, it is possible to choose an element γ_1 such that

$$\operatorname{Ad}\gamma_1 \cdot \mathfrak{a}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}}.$$

But $\operatorname{Ad}\gamma_1$ leaves $\mathfrak{a}_{\mathfrak{p}}$ pointwise fixed. Therefore, the action of γ_1 on $\mathfrak{a}_{\mathfrak{k}}$ can be regarded as coming from an element of the subgroup of W generated by those roots in P which vanish on $\mathfrak{a}_{\mathfrak{p}}$. That is, the action of $\operatorname{Ad}\gamma_1$ on $\mathfrak{a}_{\mathfrak{k}}$ is the same as for some element in W_1. Therefore, we can choose a new element γ, in the same component of M, that leaves $\mathfrak{a}_{\mathfrak{k}}$ pointwise fixed. This means that γ is in A_I. Therefore, A_I has the same number of connected components as M.

As usual, let

$$\rho = \frac{1}{2}\sum_{\alpha\in P}\alpha, \qquad \rho_M = \frac{1}{2}\sum_{\alpha\in P_M}\alpha.$$

Then since G is acceptable, it is known that M^0 is also acceptable and that for any a_1 in A_I^0

$$\xi_\rho(a_1) = \xi_{\rho_M}(a_1) \tag{2.1}$$

[see **3(j)**, Lemma 30].

Let $\mathscr{E}_M$ be the set of equivalence classes of irreducible unitary representations of M. Let C be the set of irreducible characters of the group A_I (the set of characters coming from irreducible representations of A_I). For ς in C and a in A_I write $\langle\varsigma, a\rangle$ for the value of ς at a. It is clear that W_1 operates on C.

Put $\tilde{\omega}^{\mathfrak{m}} = \prod_{\alpha\in P_M} H_\alpha$ and let L_1 be the lattice of real linear functionals, μ, on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ such that ξ_μ exists. Let $L_1' = \{\mu \in L_1 : \tilde{\omega}^{\mathfrak{m}}(\mu) \neq 0\}$. Let $Z(A) = \{\gamma \in A_I : j(\gamma) \in \exp(-1)^{1/2}\mathfrak{a}_{\mathbf{R}}\}$. Then $Z(A)$ is a finite subgroup of A_I. It is known that if γ is in $Z(A)$ and m is in M^0, then γ and m commute [**3(j)**, Lemma 51]. Also $Z(A)A_I^0 = A_I$, by [**3(k)**, Lemma 20], so $Z(A)M^0 = M$. Let $Z(A)^0 = Z(A) \cap A_I^0$. $Z(A)^0$ is a central subgroup of both $Z(A)$ and M^0. Then M is the central product of M^0 and $Z(A)$ with respect to $Z(A)^0$ (see [**2**, p. 29]). Thus if $\overline{M} = M^0 \times Z(A)$ and $\overline{Z(A)}^0 = \{(\gamma, \gamma^{-1}) : \gamma \in Z(A)^0\}$ then $\overline{Z(A)}^0$ is a discrete normal subgroup of $\overline{M}$. M is isomorphic to $\overline{M}/\overline{Z(A)}^0$. Similarly, if $\overline{A_I} = A_I^0 \times Z(A)$, then A_I is isomorphic to $\overline{A_I}/\overline{Z(A)}^0$. Therefore, irreducible representations of M (or A_I) are in one-to-one correspondence with representations of $\overline{M}$ (or

A_I) which are trivial on $\overline{Z(A)}^0$. An irreducible representation of $\overline{A_I}$ is of the form $\xi_\mu \otimes \delta$, where μ is in L_1 and δ is an irreducible representation of $Z(A)$. Let C' be the set of irreducible characters ς in C that come from representations $\xi_\mu \otimes \delta$ of $\overline{M}$ for which μ is actually in L_1'. If μ and ς are so related, we shall write $\mu = \mu_\varsigma$. We would like to prove a lemma which will relate the representations in $\mathscr{E}_M$ with characters in C'.

Let σ be an arbitrary representation of M. Then

$$\sigma = \sigma_0 \times \varepsilon \tag{2.2}$$

where σ_0 and ε are irreducible representations of M^0 and $Z(A)$, respectively, such that for any γ_0 in $Z(A)^0$, $\sigma_0(\gamma_0) \otimes \varepsilon(\gamma_0^{-1})$ is the identity. $Z(A)^0$ is in the center of both M^0 and $Z(A)$ so $\sigma_0(\gamma_0)$ and $\varepsilon(\gamma_0)$ are both scalars. Therefore

$$\sigma_0(\gamma_0) = \varepsilon(\gamma_0).$$

Suppose that $\sigma_0 = \sigma_0(\mu)$ in the notation of Lemma 1. μ is a linear functional in L_1'. Then there exists an s in W_1 such that $s\mu - \rho$ is the highest weight for σ_0. Let γ_0 be an element in $Z(A)^0$. By looking at the action of $\xi_{s\mu} - \rho(\gamma_0)$ on a highest weight vector for σ_0 we see that the scalar $\sigma_0(\gamma_0)$ is equal to $\xi_{s\mu} - \rho(\gamma_0)$. Therefore

$$\varepsilon(\gamma_0) = \sigma_0(\gamma_0) = \xi_{s\mu}(\gamma_0)\xi_\rho(\gamma_0^{-1}) = \xi_\mu(s^{-1}\gamma_0)\xi_0(\gamma_0^{-1}).$$

However, γ_0 is in the center of M so $s^{-1}\gamma_0 = \gamma_0$. Therefore

$$\varepsilon(\gamma_0)\xi_\rho(\gamma_0) = \xi_\mu(\gamma_0) \tag{2.3}$$

for any γ_0 in $Z(A)^0$.

For any γ in $Z(A)$, define

$$\delta(\gamma) = \varepsilon(\gamma)\xi_\rho(\gamma). \tag{2.4}$$

This is an irreducible representation of $Z(A)$ and by (2.3), $\xi_\mu \otimes \delta$ can be regarded as in irreducible representation of A_I. Let

$$\langle \varsigma, \gamma a_0 \rangle = \varsigma_\mu(a_0) \cdot \operatorname{tr} \delta(\gamma) \tag{2.5}$$

for a_0 in A_I^0, γ in $Z(A)$. ς is an element in C' and $\mu = \mu_\varsigma$. Therefore, given a σ in $\mathscr{E}_M$, we have constructed an element ς in C'. We write $\sigma = \sigma(\varsigma)$.

Conversely, let us start with an element in C'. By working backward we can show that there is a unique element σ in $\mathscr{E}_M$ such that $\sigma = \sigma(\varsigma)$.

Suppose that a_0 is a regular element of A_I^0 and γ is in $Z(A)$. We wish to compute the trace of $\sigma(a_0\gamma)$. Define

$$\Delta_M(a_0) = \xi_\rho(a_0) \cdot \prod_{\alpha \in P_M} (1 - \xi_\alpha(a_0^{-1})).$$

In the above notation

$$\operatorname{tr} \sigma(a_0\gamma) = \operatorname{tr} \sigma_0(a_0) \cdot \operatorname{tr} \varepsilon(\gamma).$$

But from Lemma 1,

$$\operatorname{tr}\sigma(a_0) = (\operatorname{sign}\tilde{\omega}^{\mathfrak{m}}(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s)\xi_{s\mu}(a_0) \right).$$

Therefore, the trace of $\sigma(a_0\gamma)$ is equal to

$$(\operatorname{sign}\tilde{\omega}^{\mathfrak{m}}(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s)\langle s\varsigma, a_0\gamma\rangle \right) \xi_\rho(\gamma^{-1}).$$

Now it is easy to show that if $\gamma_{\mathfrak{c}}$ is in $j(Z(A))$ then $(\gamma_{\mathfrak{c}})^2 = 1$. Therefore if γ is in $Z(A)$, $\xi_\rho(\gamma) = \xi_\rho(\gamma)^{-1}$. For future convenience, we rewrite the trace of $\sigma(a_0\gamma)$ as

$$(\operatorname{sign}\tilde{\omega}^{\mathfrak{m}}(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s)\langle s\varsigma, a_0\gamma\rangle \right) \xi_\rho(\gamma). \tag{2.6}$$

LEMMA 2. *There is a map $\varsigma \to \sigma(\varsigma)$ from C' onto $\mathcal{E}_M$. $\sigma(\varsigma_1) = \sigma(\varsigma_2)$ if and only if $s\varsigma_1 = \varsigma_2$ for some s in W_1. If a_0 is a regular element in A_I^0 and γ is in $Z(A)$ then the trace of $\sigma(\varsigma)(a_0\gamma)$ equals*

$$(\operatorname{sign}\tilde{\omega}^{\mathfrak{m}}(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s)\langle s\varsigma, a_0\gamma\rangle \right) \cdot \xi_\rho(\gamma).$$

Also, there exists a constant C_M, independent of ς, such that

$$\dim\sigma(\varsigma) = C_M \cdot |\tilde{\omega}^{\mathfrak{m}}(\mu_\varsigma)| \cdot \dim\varsigma.$$

($\dim\varsigma$ *means the dimension of the representation of A_I of which ς is the character.*)

PROOF. The dimension formula follows from Lemma 1. All other statements in the lemma follow from the above discussion. □

Let us say that the linear functional μ_ς is associated with σ if $\sigma = \sigma(\varsigma)$, in the above notation. For any σ in $\mathcal{E}_M$ there are exactly $[W_1]$ associated real linear functionals on $\mathfrak{a}_{\mathfrak{k}}$.

Now with B there is associated a discrete series of unitary representations of G. With A there is associated a continuous series. We shall describe these.

For the discrete series there is a formal analogy with Lemma 1. Let Σ be the set of positive roots of $(\mathfrak{g}_{\mathfrak{c}}, \mathfrak{b}_{\mathfrak{c}})$ relative to some order. For any α in Σ define H_α in $(-1)^{1/2}\mathfrak{b}$ by the formula

$$B(H_\alpha, H) = \alpha(H)$$

for any H in $\mathfrak{b}_{\mathfrak{c}}$. Put $\tilde{\omega}^{\mathfrak{b}} = \prod_{\alpha\in\Sigma} H_\alpha$. Let L be the lattice of real linear functionals, λ, on $(-1)^{1/2}\mathfrak{b}$ such that ξ_λ exists. Let $L' = \{\lambda \in L : \tilde{\omega}^{\mathfrak{b}}(\lambda) \neq 0\}$. Let $N(B)$ be the normalizer of B in G. Define

$$W_G = W_{G,\mathfrak{b}} = N(B)/B.$$

This is a finite group. It acts on B and therefore on L.

An irreducible representation π of G on a Hilbert space $\mathscr{H}$ is said to be square-integrable if there exist nonzero vectors Φ_1, Φ_2 if $\mathscr{H}$ such that $(\Phi_1, \pi(x)\Phi_2)$ is a square-integrable function of x. If π and π' are square-integrable representations on $\mathscr{H}$ and $\mathscr{H}'$ and if π and π' are not unitarily equivalent, then for Φ_1, Φ_2 in $\mathscr{H}$ and Φ_1', Φ_2' in $\mathscr{H}'$,

$$\int_G (\Phi_1, \pi(x)\Phi_2)(\pi'(x)\Phi_2', \Phi_1')\,dx = 0. \tag{2.7}$$

On the other hand, there is a number $\beta(\pi)$, the formal degree of π, such that for every Φ_1, Φ_2, Ψ_1, Ψ_2 and $\mathscr{H}$,

$$\int_G (\Phi_1, \pi(x)\Phi_2)(\pi(x)\Psi_2, \Psi_1)\,dx = \beta(\pi)^{-1}(\Phi_1, \Psi_1)(\Psi_2, \Phi_2). \tag{2.8}$$

These are the Schur orthogonality relations on G. They are proved in [**3(d)**, Theorem 1].

Let $\mathscr{E}_d$ be the set of unitary equivalence classes of square-integrable representations of G. Harish-Chandra gives a map $\lambda \to \omega(\lambda)$ from L' onto $\mathscr{E}_d$ [see **3(l)**, Theorem 16]. $\omega(\lambda_1) = \omega(\lambda_2)$ if and only if there is an s in W_G such that $s\lambda_1 = \lambda_2$. Finally, there is a constant C_G, independent of λ, such that

$$\beta(\omega(\lambda)) = C_G |\tilde{\omega}^{\mathfrak{b}}(\lambda)|.$$

LEMMA 3. *$\{\beta(\omega) : \omega \in \mathscr{E}_d\}$ is bounded away from zero.*

PROOF. It is clearly enough to show that for any α in Σ, $\{\lambda(H_\alpha)\}_{\lambda \in L'}$ is bounded away from zero. Let $\tilde{L}$ be the lattice of real linear functionals on $(-1)^{1/2}\mathfrak{b}$ generated by the roots. Then it is known that $L/\tilde{L}$ is isomorphic to the center of G, which is finite. It is also known that $\{\tilde{\lambda}(H_\alpha)\}_{\tilde{\lambda} \in \tilde{L}}$ is a lattice in $\mathbf{R}$. Therefore $\{\lambda(H_\alpha)\}_{\lambda \in L}$ is also a lattice in $\mathbf{R}$. But if λ is in L', $\lambda(H_\alpha) \neq 0$, so the lemma follows. $\square$

Now we shall describe the continuous series. There is a linear functional μ_0 from $\mathfrak{a}_\mathfrak{p}$ to $\mathbf{R}$ such that the restriction of any root in P_+ to $\mathfrak{a}_\mathfrak{p}$ is either μ_0 or $2\mu_0$. Fix H_0 in $\mathfrak{a}_\mathfrak{p}$ so that $\mu_0(H_0) = 1$. Extend the definition of μ_0 to $\mathfrak{a}$ by letting it equal zero on $\mathfrak{a}_\mathfrak{k}$.

Let $\mathfrak{n}_\mathbf{c} = \sum_{\alpha \in P_+} \mathbf{C}X_\alpha$, where for any α in P, X_α is a fixed root vector. Let $\mathfrak{n} = \mathfrak{n}_\mathbf{c} \cap \mathfrak{g}$. Let N be the analytic subgroup of G corresponding to $\mathfrak{n}$. It is well known (see [**4**, p. 373]) that the map

$$(k, a, n) \to kan, \qquad k \in K,\ a \in A_\mathfrak{p},\ n \in N,$$

is a diffeomorphism of $K \times A_\mathfrak{p} \times N$ with G. For f in $C_0^\infty(G)$,

$$\int_G f(x)\,dx = \int_{K \times A_\mathfrak{p} \times N} f(kan) e^{2\rho(\log a)}\,dk\,da\,dn \tag{2.9}$$

for a suitable normalization of the Haar measure dx. If $x = kan$, write $K(x) = k$ and $H(x) = \log a$.

It is clear that $P = MA_{\mathfrak{p}}N$ is a subgroup of G. If σ in $\mathscr{E}_M$ acts on a finite dimensional Hilbert space V_σ, and if Λ is in $\mathbf{R}$, then the map σ_Λ from P into $\operatorname{End}(V_\sigma)$ given by

$$\sigma_\Lambda(m \cdot \exp tH_0 \cdot n) = \sigma(m)e^{-i\Lambda t}, \qquad m \in M,\ n \in N,\ t \in \mathbf{R}$$

is an irreducible unitary representation of P. (We shall sometimes write i instead of $(-1)^{1/2}$.) Let $\pi_{\sigma,\Lambda}$ be the unitary representation of G on the Hilbert space $\mathscr{H}_{\sigma,\Lambda}$ obtained by inducing σ_Λ from P to G.

Then $\mathscr{H}_{\sigma,\Lambda}$ is the set of functions Φ from G into V_σ such that

$$\Phi(x\xi^{-1}) = \sigma_\Lambda(\xi)\Phi(x), \qquad x \in G,\ \xi \in P, \tag{2.10}$$

(2.11) $\Phi(k)$ is a Borel function on K,

$$\int_K \|\Phi(k)\|^2\, dk < \infty. \tag{2.12}$$

The inner product on $\mathscr{H}_{\sigma,\Lambda}$ is given by

$$(\Phi, \Psi) = \int_K (\Phi(k), \Psi(k))_{V_\sigma}\, dk, \qquad \Phi, \Psi \in \mathscr{H}_{\sigma,\Lambda},$$

where $(\ ,\ )_{V_\sigma}$ is the inner product in V_σ. If Φ is in $\mathscr{H}_{\sigma,\Lambda}$, $\pi_{\sigma,\Lambda}(y)\Phi$ is given by

$$(\pi_{\sigma,\Lambda}(y)\Phi)(x) = \Phi(y^{-1}x)e^{-\rho(H(y^{-1}x))+\rho(H(x))}, \qquad x, y \in G. \tag{2.13}$$

For any real Λ, and any Φ in $\mathscr{H}_{\sigma,\Lambda}$ we can define a function $\tilde{\Phi}$ from K to V_σ by restricting Φ to K. This identifies $\mathscr{H}_{\sigma,\Lambda}$ with a Hilbert space, $\mathscr{H}_\sigma$, of square-integrable functions from K into V_σ. $\mathscr{H}_\sigma$ is independent of Λ. In fact, if π_σ is the representation of K obtained by inducing σ to K, $\mathscr{H}_\sigma$ is the Hilbert space on which π_σ acts. The above equivalence between $\mathscr{H}_\sigma$ and $\mathscr{H}_{\sigma,\Lambda}$ gives an intertwining operator between π_σ and $\pi_{\sigma,\Lambda}|_K$, the restriction of $\pi_{\sigma,\Lambda}$ to K.

Let M' be the normalizer of $\mathfrak{a}_{\mathfrak{p}}$ in K. M is a normal subgroup of M'. M'/M is a group consisting of two elements, $\{1, \delta\}$ say. δ acts on $\mathfrak{a}_{\mathfrak{p}}$ by reflection. δ also induces an automorphism of M, modulo the group of inner automorphisms. Therefore δ defines a bijection.

$$\delta : \sigma \to \sigma'$$

of $\mathscr{E}_M$ onto itself. If we let δ act on P, we can transform the representation σ_Λ into the representation $(\sigma')_{-\Lambda}$. Now, if Λ is real and σ is in $\mathscr{E}_M$, it is known that the representation $\pi_{\sigma,\Lambda}$ is equivalent to $\pi_{\sigma',-\Lambda}$. Furthermore, the representations $\{\pi_{\sigma,\Lambda}\}$, $\sigma \in \mathscr{E}_M$, $\Lambda > 0$ are all irreducible and inequivalent [see **1**, Theorem 7; **2**].

For each σ in $\mathscr{E}_M$ and $\Lambda \neq 0$, let $N_\sigma(\Lambda)$ be a fixed unitary intertwining operator between the representations $\pi_{\sigma,\Lambda}$ and $\pi_{\sigma',-\Lambda}$. Then

$$N_\sigma(\Lambda)\pi_{\sigma,\Lambda}(x)N_\sigma(\Lambda)^{-1} = \pi_{\sigma',-\Lambda}(x), \qquad x \in G.$$

Notice that since $\pi_{\sigma,\Lambda}$ is irreducible,

$$N_{\sigma'}(-\Lambda) = N_\sigma(\Lambda)^{-1}. \tag{2.14}$$

It will be convenient to assign a positive real number to any equivalence class of representations in either $\mathscr{E}_d$ or $\mathscr{E}_M$. If ω is in $\mathscr{E}_d$, choose λ in L' such that $\omega = \omega(\lambda)$. The Killing form, B, of $\mathfrak{g}_c$ can be regarded as a positive definite form on either $(-1)^{1/2}\mathfrak{b}$ or its real dual space. Then put

$$|\omega|^2 = B(\lambda, \lambda).$$

Since W_G acts on $(-1)^{1/2}\mathfrak{b}$ as a group of isometries under the Killing form, $|\omega|$ is well defined. Similarly, for σ in $\mathscr{E}_M$, we define

$$|\sigma|^2 = B(\mu_\sigma, \mu_\sigma)$$

where μ_σ is any real linear functional on $(-1)^{1/2}\mathfrak{a}_\mathfrak{k}$ associated with σ. $|\sigma|$ is well defined by the above argument.

Let $\mathscr{E}_K$ be the set of unitary equivalence classes of irreducible representations of K. Let $\mathfrak{h}$ be the subspace of $\mathfrak{k}$ which is equal to either $\mathfrak{b}$ or $\mathfrak{a}_\mathfrak{k}$, depending on whether we are in Case I or Case II. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$. In either case, we have already ordered the dual space of $\mathfrak{h}$. K is acceptable by assumption, so the representations in $\mathscr{E}_K$ can be indexed by certain real linear functionals on $(-1)^{1/2}\mathfrak{h}$ as in Lemma 1. If τ is in $\mathscr{E}_K$ and $\tau = \tau(\mu)$ for some real linear functional μ on $(-1)^{1/2}\mathfrak{h}$, then we write

$$|\tau|^2 = B(\mu, \mu).$$

$|\tau|$ is well defined.

3. Plancherel formula for $L^2(G)$. In order to put the Plancherel formula for G into the form we want, we must discuss characters of unitary representations of G. To do this we must introduce some more notation of Harish-Chandra.

For t in $\mathbf{R}$, put $h_t = \exp tH_0$. For g in $C_0^\infty(M^0 A_\mathfrak{p})$, write

$$F_g^M(a_0 h_t) = \Delta_M(a_0) \cdot \int_{M^0/A_I^0} g(m^{*-1} a_0 h_t m^*)\, dm^*$$

for a_0 in A_I^0 and $a_0 h_t$ a regular element in A. Here dm^* is the invariant measure on the homogeneous space M^0/A_I^0. It is known that there exists a constant $c_1 > 0$ such that for any g in $C_0^\infty(M^0 A_\mathfrak{p})$

$$\int_{M^0 \times \mathbf{R}} g(m_0 h_t)\, dm_0\, dt = c_1 \int_{A_I^0 \times \mathbf{R}} \overline{\Delta_M(a_0)} \cdot F_g^M(a_0 h_t)\, da_0\, dt \tag{3.1}$$

(see [**3(j)**, Lemma 41]).

For a in A_I and ah_t a regular element in A write

$$\Delta(ah_t) = \xi_\rho(ah_t) \cdot \prod_{\alpha \in P} (1 - \xi_\alpha(ah_t)^{-1}),$$

$$\varepsilon_{\mathbf{R}}(ah_t) = 1 \quad \text{if } t > 0, \qquad = -1 \quad \text{if } t < 0.$$

If f is in $C_0^\infty(G)$ write

$$F_f(ah_t) = \varepsilon_R(ah_t) \cdot \Delta(ah_t) \cdot \int_{G^*} f(x^{*-1} a h_t x^*)\, dx^*.$$

(G^* is the homogeneous space G/A_0 where A_0 is the center of A. Let dx^* be the G invariant measure on G^*). It is clear that

$$F_f(sah_t) = \varepsilon(s)F_f(ah_t), \qquad s \in W_1. \tag{3.2}$$

It is known that if f is in $C_0^\infty(G)$, then F_f extends to an infinitely differentiable function on A (see [**3(f)**, Lemma 40]). Furthermore, F_f has compact support in A [**3(f)**, Theorem 2].

Let $\tilde{\omega} = \tilde{\omega}^{\mathfrak{a}} = \prod_{\alpha \in P} H_\alpha$. Let

$$q = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{k} - \operatorname{rank} \mathfrak{g} + \operatorname{rank} \mathfrak{k}).$$

It is known that q is an integer. If we are in Case II, the Cartan subgroup A is fundamental, in the terminology of [**3(f)**, p. 759]. Then Harish-Chandra's limit formula applies to F_f [**3(f)**, Theorem 4]. Namely, there is a positive constant c such that for any f in $C_0^\infty(G)$

$$cf(1) = (-1)^q F_f(1; \tilde{\omega}). \tag{3.3}$$

Here $\tilde{\omega}$ is to be regarded as a differential operator on A.

For f in $C_0^\infty(G)$ define a function g_f in $C_0^\infty(MA_{\mathfrak{p}})$ by

$$g_f(mh_t) = e^{\rho(tH_0)} \int_N \int_K f(kmh_t nk^{-1})\, dk\, dn, \qquad m \in M,\ t \in \mathbf{R}.$$

For γ in $Z(A)$ and m_0 in M^0 put

$$g_{f,\gamma}(m_0 h_t) = g_f(\gamma m_0 h_t).$$

Then in [**3(j)**, Lemma 52] it is shown that there is a constant $c_2 > 0$ such that

$$F_f(\gamma a_0 h_t) = c_2 \cdot \xi_\rho(\gamma) F^M_{g_{f,\gamma}}(a_0 h_t). \tag{3.4}$$

While we are at it, we shall state another Jacobian formula which we shall need later in the paper (see [**4**, p. 381, Proposition 1.17]). The map from $K \times \mathfrak{a}_{\mathfrak{p}}^+ \times K$ into G given by

$$(k_1, tH_0, k_2) \to k_1 \cdot \exp tH_0 \cdot k_2$$

is a diffeomorphism onto G. (We write $\mathfrak{a}_{\mathfrak{p}}^+ = \{tH_0 : t > 0\}$.) Furthermore, there is a constant $c > 0$ such that for any f in $C_0^\infty(G)$

$$\begin{aligned} \int_G f(x)\, dx &= c \int_0^\infty \int_{K \times K} f(k_1 \cdot \exp tH_0 \cdot k_2) |D(t)|\, dk_1\, dk_2\, dt \\ &= \frac{c}{2} \int_{-\infty}^\infty \int_{K \times K} f(k_1 \cdot \exp tH_0 \cdot k_2) |D(t)|\, dk_1\, dk_2\, dt. \end{aligned} \tag{3.5}$$

Here $D(t) = (e^t - e^{-t})^{r_1} \cdot (e^{2t} - e^{-2t})^{r_2}$, where r_1 and r_2 are the number of roots in P_+ which, when restricted to $\mathfrak{a}_{\mathfrak{p}}$, are respectively equal to μ_0 and $2\mu_0$.

Let π be an irreducible unitary representation of G on the Hilbert space $\mathscr{H}$. Let f be a function in $C_0^\infty(G)$. It is known that the operator

$$\pi(f) = \int_G f(x)\pi(x)\, dx$$

is of trace class. The map

$$f \to \operatorname{tr} \pi(f)$$

is a distribution on $C_0^\infty(G)$ (see [**3(c)**, §5]). This distribution is called the character of π.

If σ is in $\mathscr{E}_M$, and Λ is in $\mathbf{R}$, let $\Theta_{\sigma,\Lambda}$ be the character of the representation $\pi_{\sigma,\Lambda}$. Let $m = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$. Choose ς in C' and the associated μ_ς in L_1 such that $\sigma = \sigma(\varsigma)$ as in Lemma 2.

THEOREM 1. *There exists a constant $c_0 > 0$ such that for every f in $C_0^\infty(G)$, $\Theta_{\sigma,\Lambda}(f)$ is equal to*

$$c_0(-1)^{m+\iota}(\operatorname{sign} \tilde{\omega}^{\mathrm{m}}(\mu_\varsigma)) \cdot \int_{A_I \times \mathbf{R}} F_f(ah_t)\langle \varsigma, a\rangle e^{-i\Lambda t}\, da\, dt,$$

where ι equals 1 *or* 0 *depending on whether we are in Case* I *or Case* II.

PROOF. Let A be the operator

$$\int_G f(x)\pi_{\sigma,\Lambda}(x)\, dx$$

on $\mathscr{H}_{\sigma,\Lambda}$. We want to compute the trace of A. $\operatorname{tr} A$ is equal to $\overline{\operatorname{tr} A^*}$ where A^* is the adjoint operator of A and the bar denotes complex conjugation. If Φ is in $\mathscr{H}_{\sigma,\Lambda}$ and k_1 is in K

$$\begin{aligned}(A^*\Phi)(k_1) &= \left(\int_G \overline{f(x)}\pi(x^{-1})\, dx \cdot \Phi\right)(k_1) \\ &= \int_G \overline{f(x)}\Phi(xk_1)e^{-\rho(H(xk_1))}\, dx \\ &= \int_G \overline{f(xk_1^{-1})}\Phi(x)e^{-\rho(H(x))}\, dx.\end{aligned}$$

Assume that the Haar measure on $A_\mathfrak{p}$ has been normalized so that $dh_t = dt$. Then by (2.9) the above integral equals

$$\int_{K\times\mathbf{R}\times N} \overline{f(kh_tnk_1^{-1})}e^{i\Lambda t}e^{\rho(tH_0)}\Phi(k)\, dk\, dt\, dn.$$

In this integral, substitute km for k and integrate with respect to M. Then $(A^*\Phi)(k_1)$ equals

$$\int_{K\times M\times\mathbf{R}\times N} \overline{f(kmh_tnk_1^{-1})}\sigma(m^{-1})e^{i\Lambda t}e^{\rho(tH_0)}\Phi(k)\, dm\, dt\, dn\, dk.$$

Now to deal further with this expression we consider the principal fiber bundle

$$M \to K \to K/M.$$

The map $m \to \sigma(m^{-1})$ defines a complex vector bundle E_σ over K/M with fiber V_σ, the space on which σ acts. Let $F(k_1, k)$ be the function

$$\int_{M\times\mathbf{R}\times N} \overline{f(kmh_tnk_1^{-1})}\sigma(m^{-1})e^{i\Lambda t}e^{\rho(tH_0)}\, dm\, dt\, dn.$$

Now it is easy to check that M normalizes N and that for fixed m in M the measures dn and $d(mnm^{-1})$ on N are equal. Then for $\overline{m_1}$, $\overline{m}$ in M,

$$F(k_1\overline{m_1}, k\overline{m}) = \sigma(\overline{m}_1^{-1})F(k_1, k)\sigma(\overline{m}).$$

Therefore $F(k_1, k)$ can be regarded as a section of $E_\sigma \boxtimes E_\sigma^*$, where E_σ^* is the adjoint bundle of E_σ and $E_\sigma \boxtimes E_\sigma^*$ is the exterior tensor product of E_σ and E_σ^*, a bundle with base space $K/M \times K/M$ and fiber $V_\sigma \otimes V_\sigma^*$.

Now there is a natural equivalence between $\mathscr{H}_{\sigma,\Lambda}$ and the space $\mathscr{H}_\sigma$ defined earlier. However, $\mathscr{H}_\sigma$ is the space of square-integrable sections of E_σ with respect to a K-invariant measure on K/M. $F(k_1, k)$ can be regarded as the kernel of the linear operator A^* on this space. Then for any Φ on $\mathscr{H}_\sigma$

$$(A^*\Phi)(k_1) = \int_K F(k_1, k)\Phi(k)\, dk.$$

To evaluate the trace of A^* we appeal to the following lemma.

LEMMA 4. *Let X be a compact infinitely differentiable manifold of dimension n. Let dx be a positive nowhere-vanishing differentiable n-form on X. If $E \to X$ is a differentiable Hilbert bundle of fiber dimension s, let $L^2(E)$ be the Hilbert space of square-integrable sections of E. If $F(x_1, x)$ is a continuous section of $E \boxtimes E^*$, $F(x_1, x)$ defines a bounded linear operator F on $L^2(E)$ in the obvious manner. Then if $F(x_1, x)$ is differentiable in both variables, F is of trace class. Furthermore*

$$\operatorname{tr} F = \int_X (\operatorname{tr} F(x, x))\, dx.$$

PROOF. Let T be the closed unit n-cube with opposite sides identified. T is an n-torus and there is a canonical n-form dt on T. Let

$$S = \{t \in \mathbf{R}^n : |t| < 1\}.$$

S is an open subset of T.

Choose a finite differentiable partition of unity $\{\Psi_\alpha\}_{\alpha \in I}$ and a collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of X such that the support of Ψ_α is contained in U_α. We assume that for every (α, β) in $I \times I$ there is a diffeomorphism $\lambda_{\alpha\beta}$ from $U_\alpha \cup U_\beta$ onto $S_{\alpha\beta}$, an open subset of S. It can be seen that with no loss of generality we may also assume that

(i) $\lambda_{\alpha\beta}^*(dt) = dx$.

(ii) If $E_{\alpha\beta}$ is the restriction of E to $U_\alpha \cup U_\beta$, then $E_{\alpha\beta}$ is trivial.

(iii) The map $\lambda_{\alpha\beta}$ lifts to a bundle map

$$\Lambda_{\alpha\beta} : E_{\alpha\beta} \to S_{\alpha\beta} \times \mathbf{R}^s$$

which is an isomorphism between Hilbert bundles preserving the inner product on each fiber. (We assume that $\mathbf{R}^s$ is equipped with the natural scalar product.)

Let $F_{\alpha\beta}$ be the integral operator on $L^2(E)$ with kernel

$$F_{\alpha\beta}(x_1, x) = \Psi_\alpha(x_1)F(x_1, x)\Psi_\beta(x).$$

It is clear that if each $F_{\alpha\beta}$ is of trace class then so is F. In that case $\operatorname{tr} F = \Sigma_{\alpha\beta} \operatorname{tr} F_{\alpha\beta}$. Furthermore

$$\begin{aligned}\int_X (\operatorname{tr} F(x,x))\, dx &= \int_X \left(\sum_\alpha \Psi_\alpha(x)\right) (\operatorname{tr} F(x,x)) \left(\sum_\beta \Psi_\beta(x)\right) dx \\ &= \sum_{\alpha\beta} \int_X (\operatorname{tr} F_{\alpha\beta}(x,x))\, dx.\end{aligned}$$

Therefore it is enough to prove our lemma for the operators $F_{\alpha\beta}$.

$L^2(E_{\alpha\beta})$ is a closed subspace of $L^2(E)$. It is an invariant subspace for the operator $F_{\alpha\beta}$. $F_{\alpha\beta}$ equals zero on the complement of $L^2(E_{\alpha\beta})$ in $L^2(E)$, so the trace of $F_{\alpha\beta}$ is equal to the trace of the restriction of $F_{\alpha\beta}$ to $L^2(E_{\alpha\beta})$. Let $\mathscr{E}(\mathbf{R}^s)$ be the space of linear transformations of $\mathbf{R}^s$. Use the map $\Lambda_{\alpha\beta}$ to transform $F_{\alpha\beta}(x_1, x)$ into a section $R(t_1, t)$ of $(S_{\alpha\beta} \times S_{\alpha\beta}) \times \mathscr{E}(\mathbf{R}^s)$. Then we can regard $R(t_1,t)$ as an element in $C^\infty(T\times T) \times \mathscr{E}(\mathbf{R}^s)$. We have reduced our lemma to the case where $X = T$, $dx = dt$, $E = T \times \mathbf{R}^s$ and $F = R$.

Let $\{\phi_1(t), \phi_2(t), \dots\}$ be an orthonormal basis $L^2(T)\otimes\mathbf{R}^s$, consisting of functions of the form

$$e^{2\pi i(\nu,t)} \otimes v.$$

Here ν will be an n-tuple of integers and v will be a unit vector in $\mathbf{R}^s$. Let

$$r_{ij} = \int_{T\times T} (R(t_1,t)\phi_i(t), \phi_j(t_1))\, dt\, dt_1. \tag{3.6}$$

The above inner product is of course in $\mathbf{R}^s$. Since $R(t_1,t)$ is differentiable, we can show from the harmonic analysis of the group $T\times T$ that if m_1, m_2 are any positive integers,

$$\sup_{ij} |r_{ij}|(1+i)^{m_1}(1+j)^{m_2} < \infty. \tag{3.7}$$

This shows that R is of trace class.

If v is in $\mathbf{R}^s$, then from (3.6) we can show that for any t_1, t in T

$$R(t_1,t)v = \sum_{ij} r_{ij}(\phi_i(t), v)\phi_j(t_1).$$

Therefore

$$\operatorname{tr} R(t_1,t) = \sum_{ij} r_{ij}(\phi_i(t), \phi_j(t_1)). \tag{3.8}$$

We now compute the trace of the operator R.

$$\begin{aligned}\operatorname{tr} R = \sum_i r_{ii} &= \sum_i \int_T r_{ii}(\phi_i(t), \phi_i(t))\, dt \\ &= \sum_{ij} \int_T r_{ij}(\phi_i(t), \phi_j(t))\, dt.\end{aligned}$$

This last expression is absolutely convergent by (3.7). Therefore

$$\operatorname{tr} R = \int_T \sum_{ij} r_{ij}(\phi_i(t), \phi_j(t))\, dt.$$

By (3.8) this expression is equal to

$$\int_T (\operatorname{tr} R(t,t))\, dt.$$

This completes the proof of Lemma 4. □

Let us return to the proof of the theorem. By the lemma, $\operatorname{tr} A^*$ equals

$$\int_{K\times M\times \mathbf{R}\times N} \overline{f(kmh_t nk_1^{-1})} \cdot \operatorname{tr}\sigma(m^{-1}) e^{i\Lambda t} e^{\rho(tH_0)}\, dk\, dm\, dt\, dn.$$

Therefore

$$\begin{aligned}\operatorname{tr} A &= \overline{\operatorname{tr} A^*}\\ &= \int_{K\times M\times \mathbf{R}\times N} f(kmh_t nk^{-1})\, \overline{\operatorname{tr}\sigma(m^{-1})} e^{-i\Lambda t} e^{\rho(tH_0)}\, dk\, dm\, dt\, dn\\ &= \int_{M\times\mathbf{R}} g_f(mh_t)\cdot \operatorname{tr}\sigma(m)\cdot e^{-i\Lambda t}\, dm\, dt,\end{aligned}$$

since

$$\overline{\operatorname{tr}\sigma(m^{-1})} = \overline{\operatorname{tr}\sigma(m)^*} = \operatorname{tr}\sigma(m).$$

Let Z_A be a set of representatives of cosets of $Z(A)/Z(A)^0$. Then M is diffeomorphic with $Z_A \times M^0$. Therefore the trace of A equals

$$\sum_{\gamma\in Z_A} \int_{M^0\times\mathbf{R}} g_f(\gamma m_0 h_t)\cdot \operatorname{tr}\sigma(\gamma m_0) e^{-i\Lambda t}\, dm_0\, dt.$$

For any finite set S let $[S]$ denote the number of elements in S. Then $[P] = m$. Recall that

$$q = \tfrac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{k} - \operatorname{rank}\mathfrak{g} + \operatorname{rank}\mathfrak{k}).$$

Then

$$\begin{aligned} q &= \tfrac{1}{2}([P_+]+1) \qquad \text{in Case I,}\\ q &= \tfrac{1}{2}[P_+] \qquad \text{in Case II.}\end{aligned}$$

Therefore

$$[P_M] = [P] - [P_+] = m - 2q + \iota.$$

If a_0 is in A_I^0 then

$$\overline{\Delta_M(a_0)} = \Delta_M(a_0)(-1)^{[P_M]} = \Delta_M(a_0)(-1)^{m+\iota}.$$

Now for any m_0 in M

$$\operatorname{tr}\sigma(m_0^{-1}\gamma a_0 m_0) = \operatorname{tr}\sigma(\gamma a_0).$$

Therefore, from (3.1) we see that $\operatorname{tr} A$ equals

$$c_1 \sum_{\gamma\in Z_A} \int_{A_I^0\times\mathbf{R}} F^M_{g_f,\gamma}(a_0 h_t)\cdot \overline{\Delta_M(a_0)}\cdot \operatorname{tr}\sigma(\gamma a_0)\cdot e^{-i\Lambda t}\, da_0\, dt.$$

By Lemma 2, this equals

$$(\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma))(-1)^{m+\iota}c_1 \sum_{\gamma\in Z_A}\int_{A_I^0\times\mathbf{R}} \xi_\rho(\gamma)F^M_{g_f,\gamma}(a_0h_t) \times e^{-i\Lambda t}\left(\sum_{s\in W_1}\varepsilon(s)\langle s\varsigma,\gamma a_0\rangle\right) da_0\,dt.$$

By formula (3.4) this expression then equals

$$\begin{aligned}(\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma))(-1)^{m+\iota}\left(\frac{c_1}{c_2}\right)&\sum_{\gamma\in Z_A}\int_{A_I^0\times\mathbf{R}} F_f(\gamma a_0h_t)e^{-i\Lambda t} \times\left(\sum_{s\in W_1}\varepsilon(s)\langle s\varsigma,\gamma a_0\rangle\right) da_0\,dt\\ &= (\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma))(-1)^{m+\iota}\left(\frac{c_1}{c_2}\right)\int_{A_I\times\mathbf{R}} F_f(ah_t)e^{-i\Lambda t} \times\left(\sum_{s\in W_1}\varepsilon(s)\langle s\varsigma,a\rangle\right) da\,dt.\end{aligned}$$

Now if s is in W_1, substitute sa for a in the above expression. From (3.2) we obtain the formula

$$\operatorname{tr}A = (\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma))\cdot(-1)^{m+\iota}\cdot\left(\frac{c_1}{c_2}\right)\cdot[W_1]\cdot\int_{A_I\times\mathbf{R}} F_f(ah_t)e^{-i\Lambda t}\langle\varsigma,a\rangle\,da\,dt.$$

This proves the theorem if we let $c_0 = (c_1/c_2)[W_1]$. $\square$

For every ς in C' there is associated a unique μ_ς in L_1'. For any real Λ we write

$$\tilde{\omega}(\varsigma,\Lambda) = \tilde{\omega}(\mu_\varsigma + i\Lambda\mu_0).$$

For our discussion of the Plancherel formula it is necessary to examine this expression separately for Cases I and II. We have the formula

$$\tilde{\omega}(\varsigma,\Lambda) = \tilde{\omega}^{\mathrm{m}}(\mu_\varsigma)\cdot\prod_{\alpha\in P_+}\langle\mu_\varsigma + i\Lambda\mu_0, H_\alpha\rangle. \tag{3.9}$$

Now P_+ is the union of the positive real roots, $P_{\mathbf{R}}$, and the positive complex roots $P_{\mathbf{c}}$. Let η be the conjugation of $\mathfrak{g}_{\mathbf{c}}$ with respect to the real form $\mathfrak{g}$. η acts as a permutation of period 2 on P_+. A root in P_+ is fixed by η if and only if it is a real root, so the positive complex roots occur in pairs. Since $\dim\mathfrak{a}_{\mathfrak{p}} = 1$, there can be at most one positive real root. Now it is known that $\mathfrak{a}$ has a real root if and only if $\mathfrak{a}$ is not fundamental. Therefore there exists a real root if and only if we are in Case I.

If α is a complex root and α^η is its conjugate root, then

$$\langle\mu_\varsigma + i\Lambda\mu_0, H_\alpha\rangle\cdot\langle\mu_\varsigma + i\Lambda\mu_0, H_\alpha^\eta\rangle = -(\mu_\varsigma(H_\alpha)^2 + \Lambda^2\mu_0(H_\alpha)^2).$$

Therefore the sign of the real number

$$\prod_{\alpha\in P_{\mathbf{c}}}\langle\mu_\varsigma + i\Lambda\mu_0, H_\alpha\rangle$$

is equal to $(-1)^{[P_c]/2}$, which equals $(-1)^{q+\iota}$. Therefore

$$\begin{aligned}\tilde{\omega}(\varsigma,\Lambda)\cdot|\tilde{\omega}(\varsigma,\Lambda)|^{-1} &= i(-1)^{q+\iota}\cdot\operatorname{sign}\Lambda\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma) \quad \text{in Case I,}\\ \tilde{\omega}(\varsigma,\Lambda)\cdot|\tilde{\omega}(\varsigma,\Lambda)|^{-1} &= (-1)^{q}\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma) \quad \text{in Case II.}\end{aligned} \tag{3.10}$$

It is also clear that

$$\tilde{\omega}(\varsigma,-\Lambda) = (-1)^{\iota}\tilde{\omega}(\varsigma,\Lambda). \tag{3.11}$$

If ς is in C', choose σ in $\mathscr{E}_M$ such that $\sigma = \sigma(\varsigma)$. In §2 we defined the representation σ'. Choose ς' in C' such that $\sigma' = \sigma'(\varsigma')$. Given ς, ς' is not uniquely defined. However the expression

$$\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_{\varsigma'})\cdot\tilde{\omega}(\varsigma',\Lambda)$$

is well defined for any real Λ. Furthermore

$$\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_{\varsigma'})\cdot\tilde{\omega}(\varsigma',\Lambda) = \operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_{\varsigma})\cdot\tilde{\omega}(\varsigma,\Lambda). \tag{3.12}$$

For any ω in $\mathscr{E}_d$, let Θ_ω and $\beta(\omega)$ be the character and formal degree of ω. A formula for $\beta(\omega)$ was quoted in §2. It is clear that there is a polynomial p such that

$$\beta(\omega) \le p(|\omega|), \qquad \omega \in \mathscr{E}_d. \tag{3.13}$$

LEMMA 5. *There exists a nonnegative function $\beta(\sigma,\Lambda)$ on $\mathscr{E}_M \times \mathbf{R}$ such that for any f in $C_0^\infty(G)$,*

$$f(1) = \sum_{\omega\in\mathscr{E}_d}\beta(\omega)\Theta_\omega(f) + \sum_{\sigma\in\mathscr{E}_M}\int_0^\infty \beta(\sigma,\Lambda)\Theta_{\sigma,\Lambda}(f)\,d\Lambda.$$

In addition $\beta(\sigma,\Lambda)$ has the following properties.

(i) $\beta(\sigma,\Lambda) = \beta(\sigma,-\Lambda) = \beta(\sigma',\Lambda)$.

(ii) *For any σ in $\mathscr{E}_M$, $\beta(\sigma,\Lambda)$ is the restriction to $\mathbf{R}$ of a meromorphic function on $\mathbf{C}$ with no real poles.*

(iii) *If σ is in $\mathscr{E}_M$ and $\Lambda \ne 0$, then $\beta(\sigma,\Lambda) \ne 0$.*

(iv) *For every $r > 0$, there are polynomials p_1, p_2 such that for σ in $\mathscr{E}_M$, Λ in $\mathbf{R}$,*

$$\left|\left(\frac{d}{d\Lambda}\right)^r \beta(\sigma,\Lambda)\right| \le p_1(|\sigma|)\cdot p_2(|\Lambda|).$$

PROOF. We deal with Case I first. Although the lemma is true in general for Case I, Harish-Chandra in [**3(k)**, §24] proves it only in case j is one-to-one; that is, $G \,\overline{\subset}\, G_{\mathbf{c}}$. We shall content ourselves with dealing with this situation.

Let $C^{\pm} = \{\varsigma \in C\colon \langle\varsigma, \exp(-1)^{1/2}\pi H_0\rangle = \pm 1\}$. Since $(\exp(-1)^{1/2}\pi H_0)^2 = 1$, any element in C is in either C^+ or C^-. Then in [**3(k)**, Lemma 56], Harish-Chandra shows that there is a constant, which he writes as c_A/cc_B, such that for every f in $C_0^\infty(G)$

$$f(1) = \sum_{\omega\in\mathscr{E}_d} \beta(\omega)\Theta_\omega(f) - (c_A/cc_B)\cdot i$$

$$\cdot(-1)^{m+q}\left\{\sum_{\varsigma\in C^+}\int_0^\infty \coth\frac{\pi\Lambda}{2}\cdot\tilde{\omega}(\varsigma,\Lambda)\left(\int_{A_I\times\mathbf{R}} F_f(ah_t)\langle\varsigma,a\rangle\cdot e^{i\Lambda t}\,da\,dt\right)d\Lambda\right.$$

$$\left.+\sum_{\varsigma\in C^-}\int_0^\infty \tanh\frac{\pi\Lambda}{2}\cdot\tilde{\omega}(\varsigma,\Lambda)\left(\int_{A_I\times\mathbf{R}} F_f(ah_t)\langle\varsigma,a\rangle e^{i\Lambda t}\,da\,dt\right)d\Lambda\right\}.$$

This equals

$$\sum_{\omega\in\mathscr{E}_d} \beta(\omega)\Theta_\omega(f) - (c_A/c_0cc_B)\cdot i$$

$$\cdot(-1)^{q-1}\left\{\sum_{\varsigma\in C^+}\int_0^\infty \coth\frac{\pi\Lambda}{2}\cdot\tilde{\omega}(\varsigma,\Lambda)\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma)\cdot\Theta_{\sigma(\varsigma),-\Lambda}(f)\right.$$

$$\left.+\sum_{\varsigma\in C^-}\int_0^\infty \tanh\frac{\pi\Lambda}{2}\cdot\tilde{\omega}(\varsigma,\Lambda)\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma)\cdot\Theta_{\sigma(\varsigma),-\Lambda}(f)\right\}.$$

We then define

$$\beta(\varsigma,\Lambda) = (c_A/c_0cc_B)(-i)(-1)^{q-1}\left\{\begin{matrix}\coth\dfrac{\pi\Lambda}{2}\\ \tanh\dfrac{\pi\Lambda}{2}\end{matrix}\right.\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma), \tag{3.14}$$

where $\coth(\pi\Lambda/2)$ or $\tanh(\pi\Lambda/2)$ is used depending on whether ς is in C^+ or C^-.

Now let us deal with Case II. Then $\mathscr{E}_d$ is empty. We can use the limit formula (3.3). Therefore

$$f(1) = (1/c)(-1)^q F_f(1;\tilde{\omega}).$$

By the Fourier inversion formula on the connected abelian group $A_I\times\mathbf{R}$,

$$f(1) = \left(\frac{1}{c}\right)(-1)^q\sum_{\varsigma\in C}\int_{-\infty}^\infty\left[\int_{A_I\times\mathbf{R}} F_f(ah_t;\tilde{\omega})\langle\varsigma,a\rangle e^{i\Lambda t}\,da\,dt\right]d\Lambda.$$

Since $m=[P]$ and since F_f is in $C_0^\infty(A_I\times\mathbf{R})$, we see by integration by parts that $f(1)$ is equal to

$$\left(\frac{1}{c}\right)(-1)^{q+m}\sum_{\varsigma\in C}\int_{-\infty}^\infty\tilde{\omega}(\varsigma,\Lambda)\left[\int_{A_I\times\mathbf{R}} F_f(ah_t)\langle\varsigma,a\rangle e^{i\Lambda t}\,da\,dt\right]d\Lambda.$$

By Theorem 1 this expression equals

$$\left(\frac{1}{c_0c}\right)(-1)^q\sum_{\varsigma\in C}\int_{-\infty}^\infty\tilde{\omega}(\varsigma,\Lambda)\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma)\cdot\Theta_{\sigma(\varsigma),-\Lambda}(f)\,d\Lambda.$$

We then define

$$\beta(\varsigma,\Lambda) = (2/c_0c)(-1)^q\cdot\tilde{\omega}(\varsigma,\Lambda)\cdot\operatorname{sign}\tilde{\omega}^{\mathrm{m}}(\mu_\varsigma). \tag{3.15}$$

In either case, we see from (3.10) that $\beta(\varsigma, \Lambda)$ is nonnegative. Also from (3.11) we see that

$$\beta(\varsigma, -\Lambda) = \beta(\varsigma, \Lambda).$$

It is clear that the expression $\beta(\varsigma', \Lambda)$ is well defined. (3.12) implies the formula

$$\beta(\varsigma', \Lambda) = \beta(\varsigma, \Lambda).$$

Since for any $\Lambda \neq 0$ the representations $\pi_{\sigma',\Lambda}$ and $\pi_{\sigma,-\Lambda}$ are equivalent,

$$\Theta_{\sigma',\Lambda} = \Theta_{\sigma,-\Lambda}.$$

Therefore in either Case I or Case II we obtain the formula

$$f(1) = \sum_{\omega \in \mathscr{E}_d} \beta(\omega)\Theta_\omega + \sum_{\varsigma \in C} \int_0^\infty \beta(\varsigma, \Lambda)\Theta_{\sigma(\varsigma),\Lambda}(f)\, d\Lambda.$$

It is clear in either case that

$$\beta(s\varsigma, \Lambda) = \beta(\varsigma, \Lambda), \qquad s \in W_1.$$

Then if σ is in $\mathscr{E}_M$, choose any ς in C' such that $\sigma = \sigma(\varsigma)$. Define

$$\beta(\sigma, \gamma) = [W_1]\beta(\varsigma, \Lambda).$$

Then $\beta(\sigma, \Lambda)$ is well defined and $\beta(\sigma, \Lambda)$ satisfies the formula of the lemma.

Property (i) of the lemma follows from the above discussion. Properties (ii), (iii), and (iv) follow easily from formulae (3.14) and (3.15). □

For ω in $\mathscr{E}_d$, let π_ω be a representation in the class of ω, acting on the Hilbert space $\mathscr{H}_\omega$. Let $\mathscr{H}_2(\omega)$ be the space of Hilbert-Schmidt operators on $\mathscr{H}_\omega$ with the Hilbert-Schmidt norm $\|\cdot\|_2$. Similarly, for σ in $\mathscr{E}_M$, write $\mathscr{H}_2(\sigma)$ as the space of Hilbert-Schmidt operators on $\mathscr{H}_\sigma$.

Let $L_0^2(\hat{G})$ be the set of functions

$$a_0 \colon \mathscr{E}_d \to \bigoplus_{\omega \in \mathscr{E}_d} \mathscr{H}_2(\omega)$$

such that

(i) $a_0(\omega)$ is in $\mathscr{H}_2(\omega)$ for each ω in $\mathscr{E}_d$.

(ii) $\|a_0\|^2 = \sum_{\omega \in \mathscr{E}_d} \|a_0(\omega)\|_2^2 \beta(\omega) < \infty$.

Notice that if we are in Case II, $\mathscr{E}_d$ is empty so that $L_0^2(\hat{G})$ is empty.

Let $L_1^2(\hat{G})$ be the set of functions

$$a_1 \colon \mathscr{E}_M \times \mathbf{R} \to \bigoplus_{\sigma \in \mathscr{E}_M} \mathscr{H}_2(\sigma)$$

such that

(i) $a_1(\sigma, \Lambda)$ is in $\mathscr{H}_2(\sigma)$ for each σ in $\mathscr{E}_M$ and Λ in $\mathbf{R}$.

(ii) $a_1(\sigma', -\Lambda) = N_\sigma(\Lambda) a_1(\sigma, \Lambda) N_\sigma(\Lambda)^{-1}$, $\sigma \in \mathscr{E}_M$, $\Lambda \neq 0$.

(iii) For any σ in $\mathscr{E}_M$, $a_1(\sigma, \Lambda)$ is a Borel function of Λ.

(iv)

$$\|a_1\|^2 = \frac{1}{2} \sum_{\sigma \in \mathscr{E}_M} \int_{-\infty}^\infty \|a_1(\sigma, \Lambda)\|_2^2 \beta(\sigma, \Lambda)\, d\Lambda < \infty.$$

(In (ii) we can regard the operators $N_\sigma(\Lambda)$ as maps from $\mathscr{H}_\sigma$ to $\mathscr{H}_{\sigma'}$ if we recall the canonical isomorphisms $\mathscr{H}_{\sigma,\Lambda} \leftrightarrow \mathscr{H}_\sigma$, $\mathscr{H}_{\sigma',-\Lambda} \leftrightarrow \mathscr{H}_{\sigma'}$.)

Notice that since $N_\sigma(\Lambda)$ is unitary, condition (ii) implies that

$$\|a_1(\sigma', -\Lambda)\|_2^2 = \|a_1(\sigma, \Lambda)\|_2^2.$$

Therefore

$$\|a_1\|^2 = \sum_{\sigma \in \mathscr{E}_M} \int_0^\infty \|a_1(\sigma, \Lambda)\|_2^2 \beta(\sigma, \Lambda)\, d\Lambda. \tag{3.16}$$

$L_0^2(\hat{G})$ and $L_1^2(\hat{G})$ are Hilbert spaces. Let $L^2(\hat{G}) = L_0^2(\hat{G}) \oplus L_1^2(\hat{G})$. If f is in $C_0^\infty(G)$, define $\hat{f}$ in $L^2(\hat{G})$ by

$$\hat{f} = (\hat{f}_0(\omega), \hat{f}_1(\sigma, \Lambda)),$$

$$\hat{f}_0(\omega) = \int_G f(x)\pi_\omega(x)\, dx, \qquad \omega \in \mathscr{E}_d,$$

$$\hat{f}_1(\sigma, \Lambda) = \int_G f(x)\pi_{\sigma,\Lambda}(x)\, dx, \qquad \sigma \in \mathscr{E}_M,\ \Lambda \in \mathbf{R}.$$

(We can regard $\hat{f}_1(\sigma, \Lambda)$ as an operator on $\mathscr{H}_\sigma$.)

THEOREM 2 (PLANCHEREL FORMULA). *The map*

$$f \to \hat{f}, \qquad f \in C_0^\infty(G),$$

extends uniquely to an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

PROOF. Fix f in $C_0^\infty(G)$. Define

$$g(x) = \int_G f(y)\overline{f(x^{-1}y)}\, dy, \qquad x \in G.$$

Clearly g is in $C_0^\infty(G)$ and $g(1)$ equals $\|f\|_2^2$. If π is an irreducible unitary representation of G,

$$\begin{aligned}
\pi(g) &= \int_{G\times G} f(y)\overline{f(x^{-1}y)}\, dy \cdot \pi(x)\, dx \\
&= \int_{G\times G} f(y)\overline{f(x^{-1})} \cdot \pi(yx)\, dy\, dx \\
&= \left(\int_G f(y) \cdot \pi(y)\, dy\right)\left(\int_G f(x) \cdot \pi(x)\, dx\right)^* \\
&= \pi(f) \cdot \pi(f)^*
\end{aligned}$$

where $\pi(f)^*$ is the adjoint of $\pi(f)$. Therefore

$$\operatorname{tr} \pi(g) = \|\pi(f)\|_2^2 = \|\hat{f}(\pi)\|_2^2.$$

Therefore, applying Lemma 5 to $g(x)$ we see that

$$\|f\|_2^2 = \|\hat{f}\|^2.$$

Thus, the map $f \to \hat{f}$ is an isometry. We need only show that it is surjective.

By the Schur orthogonality relations (2.7) and (2.8), the map is onto $L_0^2(\hat{G})$. We must show that it is onto $L_1^2(\hat{G})$.

Let $\hat{\rho}_1$ be the representation of $G \times G$ on $L_1^2(\hat{G})$ given by

$$\hat{\rho}_1(x,y)a_1(\sigma,\Lambda) = \pi_{\sigma,\Lambda}(x)a_1(\sigma,\Lambda)\pi_{\sigma,\Lambda}(y^{-1})$$

for $\sigma \in \mathscr{E}_M$, $\Lambda \in \mathbf{R}$, and $(x,y) \in G \times G$. $G \times G$, being semisimple, is of type I [**3(a)**, p. 30], so $\hat{\rho}_1$ is of type I. Let $\mathbf{R}^+ = \{\Lambda \in \mathbf{R} \colon \Lambda > 0\}$, $S = \mathscr{E}_M \times \mathbf{R}^+$, and let C be the measure class on S defined by the discrete measure on $\mathscr{E}_M$ and Lebesgue measure on $\mathbf{R}^+$. $\beta(\sigma,\Lambda)$ does not vanish for any (σ,Λ) in S, and the representations $\{\pi_{\sigma,\Lambda} \times \pi_{\sigma,\Lambda} \colon (\sigma,\Lambda) \in S\}$ of $G \times G$ are all irreducible and inequivalent. $\hat{\rho}_1$ is clearly the direct integral of these representations of $G \times G$ with respect to the measure class C. Therefore $\hat{\rho}_1$ is multiplicity-free by [**6(b)**, Theorem 5]. This means that the algebra $R(\hat{\rho}_1, \hat{\rho}_1)$ of intertwining operators of $\hat{\rho}_1$ is commutative.

Let ρ be the two-sided regular representation of $G \times G$ on $L^2(G)$. Then the map

$$f \to \hat{f}_1, \qquad f \in L^2(G),$$

is an intertwining operator between ρ and $\hat{\rho}_1$. Thus if L is the closure of the set $\{\hat{f}_1 \colon f \in L^2(G)\}$, and P is the orthogonal projection of $L_1^2(\hat{G})$ onto L, then P is in $R(\hat{\rho}_1, \hat{\rho}_1)$. But since $R(\hat{\rho}_1, \hat{\rho}_1)$ is commutative, it is well known that P is of the form P_E, where E is a Borel subset of S and

$$P_E = \{a_1 \in L_1^2(\hat{G}) \colon a_1 \text{ vanishes outside } E\}.$$

To complete the proof of the surjectivity of the map $f \to \hat{f}_1$, we need only show that the complement of E in S is a null set with respect to C.

Let us assume the contrary. Then there is a σ in $\mathscr{E}_M$ and a subset R_1 of $\mathbf{R}^+$ of positive Lebesgue measure such that for any f in $C_0^\infty(G)$,

$$\hat{f}_1(\sigma,\Lambda) = 0 \quad \text{for almost all } \Lambda \text{ in } R_1.$$

Choose a τ in $\mathscr{E}_K$ for which there is a nonzero intertwining operator T between the restriction of τ to M and σ. Choose a vector ξ in the space on which τ acts such that $T\xi \neq 0$. Define

$$\Phi(k) = T(\tau(k^{-1})\xi), \qquad k \in K.$$

Then Φ is in $\mathscr{H}_\sigma$. For any f in $C_0^\infty(G)$,

$$\begin{aligned}(\hat{f}(\sigma,\Lambda)\Phi)(1) &= \left(\int_G f(x)\pi_{\sigma,\Lambda}(x)\,dx \cdot \Phi\right)(1) \\ &= \int_G f(x^{-1})\Phi(x)e^{-\rho(H(x))}\,dx.\end{aligned}$$

Then by (2.9), $(\hat{f}(\sigma,\Lambda)\Phi)(1)$ is equal to

$$\int_{K\times\mathbf{R}\times N} f(n^{-1} \cdot \exp(-tH_0) \cdot k^{-1})e^{(i\Lambda+\rho(H_0))t}\Phi(k)\,dk\,dt\,dn.$$

Let $f(n^{-1} \cdot \exp(-tH_0) \cdot k^{-1})$ equal

$$\chi(k) \cdot \alpha(t) \cdot \nu(n)$$

where $\chi(k) = (\tau(k)\xi, \xi)$ and ν is any function in $C_0^\infty(N)$ such that $\int_N \nu(n)\,dn = 1$. α is some function in $C_0^\infty(\mathbf{R})$ such that $\int_{-\infty}^{\infty} \alpha(t)e^{(i\Lambda+\rho(H_0))t}\,dt$ is not equal to zero for any Λ belonging to a subset R_2 of R_1 of positive measure. Clearly such an α exists.

For a fixed Λ_0 in R_2,

$$(\hat{f}(\sigma, \Lambda_0)\Phi)(1) = T(\xi) \cdot \int_{-\infty}^{\infty} \alpha(t)e^{(i\Lambda_0+\rho(H_0))t}\,dt.$$

This is a nonzero vector in the space on which σ acts. However, $(\hat{f}(\sigma, \Lambda_0)\Phi)(k)$ is a continuous function of k, so $(\hat{f}(\sigma, \Lambda_0)\Phi)(k)$ is nonzero for a subset of K of positive measure. Therefore $\hat{f}(\sigma, \Lambda_0)\Phi$ is a nonzero vector in $\mathscr{H}_\sigma$. This means that the operators $\hat{f}_1(\sigma, \Lambda)$ do not vanish for any Λ in R_2. We have a contradiction. The proof of Theorem 2 is now complete. □

4. Statement of Theorem 3. For x in G, define

$$\Xi(x) = \int_K e^{-\rho(H(xk))}\,dk.$$

Define a norm on $\mathfrak{g}$ by putting

$$\|X\|^2 = -B(X, \theta X), \qquad X \in \mathfrak{g},$$

where B is the Killing form on $\mathfrak{g}$. Since $G = KA_\mathfrak{p}K$ there exist a unique function σ on G such that

(i) $\sigma(k_1xk_2) = \sigma(x)$, $k_1, k_2 \in K$, $x \in G$;

(ii) $\sigma(\exp H) = \|H\|$, $H \in \mathfrak{a}_\mathfrak{p}$.

It is known that there exist numbers c, d such that for any a in $A_\mathfrak{p}^+$ $(= \{\exp tH_0 \colon t \geqslant 0\})$,

$$1 \leqslant \Xi(a)e^{\rho(\log a)} \leqslant c(1 + \sigma(a))^d \tag{4.1}$$

(see [**3(g)**, Theorem 3 and Lemma 36]). Also there is an $r_0 > 0$ such that

$$\int_G \Xi(x)^2(1 + \sigma(x))^{-r_0}\,dx = N(r_0) < \infty \tag{4.2}$$

(see [**3(l)**, Lemma 11]).

Choose δ in K such that $\delta^{-1}a\delta = a^{-1}$ for any a in $A_\mathfrak{p}$. We obtain the formulae

$$\begin{aligned} \Xi(a^{-1}) &= \Xi(\delta^{-1}a\delta) = \Xi(a), \\ \sigma(a^{-1}) &= \sigma(\delta^{-1}a\delta) = \sigma(a). \end{aligned} \tag{4.3}$$

Let $\mathscr{B}$ be the universal enveloping algebra of $\mathfrak{g}_\mathbf{c}$. We can identify $\mathscr{B}$ with the algebra of left invariant differential operators on G. Let ρ be the canonical anti-isomorphism with $\mathscr{B}$ and the algebra of right invariant differential operators on G. If g_1 and g_2 are in $\mathscr{B}$ and f is a differentiable function on G, then the actions of $\rho(g_1)$ and g_2 on f commute. We denote the resultant of this action at any x in G by $f(g_1; x; g_2)$.

Now for every g_1, g_2, in $\mathscr{B}$ and s in $\mathbf{R}$, we define a seminorm on $C^\infty(G)$ by

$$\|f\|_{g_1,g_2,s} = \sup_{x\in G} |f(g_1;x;g_2)|\Xi(x)^{-1}(1+\sigma(x))^s, \qquad f \in C^\infty(G).$$

Let $\mathscr{C}(G) = \{f \in C^\infty(G)\colon \|f\|_{g_1,g_2,s} < \infty$, for any g_1, g_2 in $\mathscr{B}$ and s in $\mathbf{R}\}$. These seminorms make $\mathscr{C}(G)$ into a Fréchet space.

Clearly

$$C_0^\infty(G) \subset \mathscr{C}(G)$$

is a continuous inclusion, and it is known that $C_0^\infty(G)$ is dense in $\mathscr{C}(G)$ [**3(l)**, Theorem 2]. Also from (4.2) we see that there is a continuous inclusion of $\mathscr{C}(G)$ into $L^2(G)$. $\mathscr{C}(G)$ is called the Schwartz space of G.

We wish to define a subspace of $L^2(\hat{G})$ which will ultimately turn out to be the image of $\mathscr{C}(G)$ under the Fourier transform map, $f \to \hat{f}$. We shall need to fix appropriate bases for the Hilbert spaces $\mathscr{H}_\omega$ and $\mathscr{H}_{\sigma,\Lambda}$.

For each ω in $\mathscr{E}_d$ let π_ω be a representation in the class of ω acting on the Hilbert space $\mathscr{H}_\omega$. We can choose an orthonormal basis

$$\{\Phi_{\tau,i} = \Phi_{\tau,i}(\omega)\}_{\tau\in\mathscr{E}_K} \tag{4.4}$$

of $\mathscr{H}_\omega$ such that $\Phi_{\tau,i}$ transforms under $\pi_\omega|_K$, the restriction of π_ω to K, according to the irreducible representation τ of K. The second subscript, i, ranges from 1 to $[\omega\colon\tau]\cdot\dim\tau$, where $[\omega\colon\tau]$ is the multiplicity of τ in $\pi_\omega|_K$. It is known that $[\omega\colon\tau] \leqslant \dim\tau$ (see [**3(b)**, Theorem 4]).

We shall construct explicit bases for the Hilbert spaces $\mathscr{H}_{\sigma,\Lambda}$. As we remarked earlier, there is a canonical intertwining operator between the representations $\pi_{\sigma,\Lambda}|_K$ and π_σ of K. Therefore we shall choose a fixed orthonormal basis for the Hilbert space $\mathscr{H}_\sigma$.

The multiplicity of τ in $\pi_{\sigma,\Lambda}|_K$ equals the multiplicity of τ in π_σ. But π_σ is just the representation σ induced to K. Therefore by the Frobenius reciprocity theorem for compact groups [**6(a)**, Theorem 8.2], these multiplicities are just equal to $[\tau\colon\sigma]$, the multiplicity of σ in $\tau|_M$ ($\tau|_M$ is the restriction of τ to M).

Fix τ in $\mathscr{E}_K$ and σ in $\mathscr{E}_M$ acting on the Hilbert spaces V_τ and V_σ of dimension t and s respectively. Let $R(\tau,\sigma)$ be the set of intertwining operators from V_τ to V_σ for $\tau|_M$ and σ. The Hilbert-Schmidt norm makes $R(\tau,\sigma)$ into a Hilbert space of dimension $[\tau\colon\sigma]$.

Now suppose T is in $R(\tau,\sigma)$. Since σ is irreducible, we can assume that there are orthonormal bases $\{\xi_1,\ldots,\xi_t\}$ and $\{\eta_1,\ldots,\eta_s\}$ of V_τ and V_σ respectively such that there is a constant c for which

$$T\xi_i = c\eta_i, \qquad i \leqslant s,$$
$$T\xi_i = 0, \qquad i > s.$$

Suppose T has been normalized such that $c = (t/s)^{1/2}$. Then

$$\begin{aligned} T\xi_i &= (t/s)^{1/2}\eta_i, \qquad i \leqslant s, \\ \|T\| &= t^{1/2}. \end{aligned} \tag{4.5}$$

Fix an element ξ of norm 1 in V_τ. Write $\tau^*(k)$ for $\tau(k^{-1})$ if k is in K. Define

$$\Phi(k) = T(\tau^*(k)\xi), \qquad k \in K.$$

Then

(i)

$$\begin{aligned}\Phi(km^{-1}) &= T(\tau(m)\tau^*(k)\xi) = \sigma(m)T(\tau^*(k)\xi)\\ &= \sigma(m)\Phi(k), \qquad m \in M,\ k \in K.\end{aligned}$$

Therefore Φ is an element in $\mathscr{H}_\sigma$.

(ii) $\|\Phi\| = 1$, because

$$\begin{aligned}(\Phi,\Phi) &= \int_K (T(\tau^*(k)\xi), T(\tau^*(k)\xi))\,dk\\ &= \int_K \sum_{ij} (T[(\tau^*(k)\xi,\xi_i)\xi_i], T[(\tau^*(k)\xi,\xi_j)\xi_j])\,dk\\ &= \sum_{i=1}^{s} \int_K ((\tau^*(k)\xi,\xi_i)\eta_i, (\tau^*(k)\xi,\xi_i)\eta_i)\,dk \cdot \left(\frac{t}{s}\right)\\ &= \sum_{i=1}^{s} \int_K (\tau^*(k)\xi,\xi_i)\overline{(\tau^*(k)\xi,\xi_i)}\,dk \cdot \left(\frac{t}{s}\right)\\ &= \left(\frac{t}{s}\right)\cdot\left(\frac{\dim\sigma}{\dim\tau}\right) \quad \text{(by the Schur orthogonality relations on } K)\\ &= 1.\end{aligned}$$

Conversely, let Φ be any unit vector in $\mathscr{H}_\sigma$ such that Φ transforms under π_σ according to τ. Then there exists a unit vector ξ in V_τ and a T in $R(\tau,\sigma)$ with $\|T\| = (\dim\tau)^{1/2}$ such that

$$\Phi(k) = T(\tau^*(k)\xi), \qquad k \in K.$$

For Φ defined as above, the vector $N_\sigma(\Lambda)\Phi$ is in $\mathscr{H}_{\sigma'}$. Clearly $N_\sigma(\Lambda)\Phi$ transforms under $\pi_{\sigma'}$ according to τ. Then there exists a unique T' in $R(\tau,\sigma')$ with $\|T'\| = (\dim\tau)^{1/2}$ such that

$$(N_\sigma(\Lambda)\Phi)(k) = T'(\tau^*(k)\xi), \qquad k \in K.$$

The map $T \to T'$ from $R(\tau,\sigma)$ into $R(\tau,\sigma')$ will be denoted $n_\sigma(\Lambda)$, so $T' = n_\sigma(\Lambda)T$. $n_\sigma(\Lambda)$ is norm-preserving and hence unitary.

Fix an orthonormal base $\{T_1,\dots,T_r\}$ of $R(\tau,\sigma)$ of elements of norm equal to $(\dim\tau)^{1/2}$. For $1 \leqslant l \leqslant r$, $1 \leqslant j \leqslant t$, and k in K, define

$$\Phi_{\tau,(l-1)t+j}(k) = T_l(\tau^*(k)\xi_j). \tag{4.6}$$

Then $\{\Phi_{\tau,i} \colon \tau \in \mathscr{E}_K,\ 1 \leqslant i \leqslant [\tau:\sigma]\dim\tau\}$ is an orthonormal base for $\mathscr{H}_\sigma$.

The bases (4.4) and (4.6) can be used to define a collection of seminorms on $L_0^2(\hat{G})$ and $L_1^2(\hat{G})$ respectively. For each triplet (p,q_1,q_2) of polynomials we define a seminorm on $L_0^2(\hat{G})$ by letting $\|a_0\|_{p,q_1,q_2}$ be the supremum over ω, (τ_1,i_1), (τ_2,i_2) of the expressions

$$|(\Phi_{\tau_1,i_1}, a_0(\omega)\Phi_{\tau_2,i_2})|p(|\omega|)q_1(|\tau_1|)q_2(|\tau_2|), \qquad a_0 \in L_0^2(\hat{G}). \tag{4.7}$$

Let $\mathscr{C}_0(\hat{G})$ be the set of all a_0 in $L_0^2(\hat{G})$ for which $\|a_0\|_{p,q_1,q_2} < \infty$ for every triplet (p, q_1, q_2).

For each set of polynomials (p_1, p_2, q_1, q_2) and each integer n define a seminorm on $L_1^2(\hat{G})$ as follows: put $\|a_1\|_{(p_1,p_2,q_1,q_2:\, n)} = \infty$ if for some σ in $\mathscr{E}_M$ and some Φ_{τ_1,i_1} and Φ_{τ_2,i_2} the function $(\Phi_{\tau_1,i_1}, a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2})$ is not n times continuously differentiable in Λ. Otherwise, let $\|a_1\|_{(p_1,p_2,q_1,q_2:\, n)}$ equal the supremum over (σ, Λ), (τ_1, i_1), (τ_2, i_2) of the expressions

$$\left|\left(\frac{d}{d\Lambda}\right)^n (\Phi_{\tau_1,i_1}, a(\sigma,\Lambda)\Phi_{\tau_2,i_2})\right| p_1(|\sigma|)p_2(|\Lambda|)q_1(|\tau_1|)q_2(|\tau_2|). \tag{4.8}$$

Let $\mathscr{C}_1(\hat{G})$ be the set of all a_1 in $L_1^2(\hat{G})$ for which $\|a_1\|_{(p_1,p_2,q_1,q_2:\, n)} < \infty$ for every set $(p_1, p_2, q_1, q_2 \colon n)$.

The above seminorms define topologies on $\mathscr{C}_0(\hat{G})$ and $\mathscr{C}_1(\hat{G})$. Define

$$\mathscr{C}(\hat{G}) = \mathscr{C}_0(\hat{G}) \oplus \mathscr{C}_1(\hat{G}).$$

$\mathscr{C}(\hat{G})$ is a Fréchet space.

THEOREM 3. *The map $f \to \hat{f}$ gives a topological isomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$.*

We shall spend most of the rest of this paper proving this theorem.

5. Spherical functions. In this section we shall define τ-spherical functions on G and develop some of their elementary properties.

A unitary double representation τ of the compact group K is a Hilbert space on which there is both a left and a right unitary K action. In addition, these actions are required to commute with each other. We denote both the left and the right action of K by τ. If τ is a unitary double representation of K on the vector space V_τ, define a representation τ' of $K \times K$ on V_τ by

$$\tau'(k_1, k_2)v = \tau(k_1)v\tau(k_2^{-1}), \qquad v \in V_\tau,\ k_1, k_2 \in K.$$

There is a one-to-one correspondence between double representations of K and representations of $K \times K$.

Suppose τ is a unitary double representation of K on the vector space V_τ. A function ϕ from G to V_τ is said to be τ-spherical if for every k_1, k_2 in K and x in G,

$$\phi(k_1 x k_2) = \tau_1(k_1)\phi(x)\tau_2(k_2).$$

We shall write $|\phi(x)|$ to indicate the norm of $\phi(x)$ in V_τ.

Suppose $f(x)$ is a continuous complex-valued function on G such that the left and right translates of f by elements in K span a finite-dimensional space of functions on G. We shall use f to define a spherical function.

Let ϕ be the function from G into $L^2(K \times K)$ defined by

$$\phi(x)(k_1, k_2) = f(k_1^{-1} x k_2^{-1}), \qquad x \in G,\ k_1, k_2 \in K.$$

Define a double K representation μ on $L^2(K \times K)$ by

$$[\mu(\bar{k}_1)u](k_1, k_2) = u(\bar{k}_1^{-1}k_1, k_2),$$
$$[u\mu(\bar{k}_2)](k_1, k_2) = u(k_1, k_2\bar{k}_2^{-1}),$$

for u in $L^2(K \times K)$ and k_1, k_2, $\bar{k}_1$, $\bar{k}_2$ in K. Let V_μ equal $sp_{x\in G}\{\phi(x)\}$, the finite-dimensional subspace of $L^2(K \times K)$ spanned by $\{\phi(x) : x \in G\}$. Then for any x in G, and $\bar{k}_1$, $\bar{k}_2$, k_1, k_2 in K,

$$\phi(\bar{k}_1 x \bar{k}_2)(k_1, k_2) = f(k_1^{-1}\bar{k}_1 x \bar{k}_2 k_2^{-1}) = f((\bar{k}_1^{-1}k_1)^{-1}x(k_2\bar{k}_2^{-1})^{-1}.$$

This expression equals

$$(\mu(\bar{k}_1)\phi(x)\mu(\bar{k}_2))(k_1, k_2).$$

Therefore ϕ is a μ-spherical function, which we shall call the μ-spherical function associated with f.

Notice that if τ is an irreducible unitary double representation of K on the finite-dimensional Hilbert space V_τ, then τ can be regarded as an irreducible representation $\tau_1 \otimes \tau_2^*$ of $K \times K$ on $V_1 \otimes V_2^*$. Here τ_1 and τ_2 are irreducible representations of K on the spaces V_1 and V_2, and τ_2^* is the dual representation of τ_2 acting on V_2^*, the dual space of V_2. We write τ as (τ_1, τ_2) and $|\tau|$ as $|\tau_1| + |\tau_2|$. Let $\mathscr{E}_K^2$ be the set of equivalence classes of irreducible unitary double representations of K.

Suppose that $f(x)$ is the function $(\Phi_1, \pi(x)\Phi_2)$ where π is a unitary representation of G on a Hilbert space $\mathscr{H}$. We assume that for $\alpha = 1$ or 2, Φ_α is a unit vector in $\mathscr{H}$ that transforms under $\pi|_K$ according to the irreducible unitary representation τ_α of K, acting on the Hilbert space V_α. Let $\tau = (\tau_1, \tau_2) \cdot \tau$ is in $\mathscr{E}_K^2$ and acts on the Hilbert space $V_\tau = V_1 \otimes V_2^*$. We shall find a formula for the spherical function ϕ associated with f. Then we shall specialize to the case where π is one of the induced representations $\pi_{\sigma,\Lambda}$ defined in §2.

Let τ_α have dimension t_α and let $\{\xi_{\alpha 1}, \ldots, \xi_{\alpha t_\alpha}\}$ be an orthonormal base for V_α, for $\alpha = 1$ or 2. Let V'_α be the subspace of $\mathscr{H}$ spanned by $\{\pi(k)\Phi_\alpha : k \in K\}$. Choose an orthonormal base $\{\Phi_{\alpha 1}, \ldots, \Phi_{\alpha t_\alpha}\}$ of V'_α such that the correspondence

$$\xi_{\alpha i} \leftrightarrow \Phi_{\alpha i}, \qquad i = 1, 2, \ldots, t_\alpha,$$

gives an intertwining operator between τ_α and $\pi|_K$ acting on the space V'_α. Define functions $e_{1i}(k_1)$ and $e_{2j}(k_2)$ as follows:

$$e_{1i}(k_1) = (\pi(k_1)\Phi_1, \Phi_{1i}), \qquad k_1 \in K,$$
$$e_{2j}(k_2) = \overline{(\pi(k_2^{-1})\Phi_2, \Phi_{2j})} = (\pi(k_2)\Phi_{2j}, \Phi_2), \qquad k_2 \in K.$$

Then

$$\phi(x)(k_1, k_2) = f(k_1^{-1}xk_2^{-1}) = (\pi(k_1)\Phi_1, \pi(x)\pi(k_2^{-1})\Phi_2).$$

This is equal to the expression

$$\sum_{ij} e_{1i}(k_1)e_{2j}(k_2)(\Phi_{1i}, \pi(x)\Phi_{2j}).$$

V_μ is the subspace of $L^2(K \times K)$ spanned by the functions $e_{1i}(k_1)e_{2j}(k_2)$. Let $\{\xi^*_{21}, \dots, \xi^*_{2t_2}\}$ be the dual basis in V_2^* to $\{\xi_{21}, \dots, \xi_{2t_2}\}$. Then for $1 \leqslant i \leqslant t_1$, $1 \leqslant j \leqslant t_2$, identify $e_{1i}(k_1)e_{2j}(k_2)$ with $(t_1t_2)^{-1/2}\xi_{1i} \otimes \xi^*_{2j}$. This gives an intertwining operator between the double representations μ and τ. Therefore, we can regard ϕ as a τ-spherical function from G to V_τ. We have the formula

$$\phi(x) = (t_1t_2)^{-1/2} \sum_{ij} \xi_{1i} \otimes \xi^*_{2j}(\Phi_{1i}, \pi(x)\Phi_{2j}), \qquad x \in G. \tag{5.1}$$

Now suppose that π is one of the representations $\pi_{\sigma,\Lambda}$, for σ in $\mathscr{E}_M$ and Λ in $\mathbf{R}$. Recall that $R(\tau_\alpha, \sigma)$ was the space of intertwining operators between τ_α and σ. Fix T_α in $R(\tau_\alpha, \sigma)$ such that

$$\|T_\alpha\|^2 = \dim \tau_\alpha = t_\alpha, \qquad \alpha = 1, 2.$$

Suppose that ξ_1 and ξ_2 are unit vectors in V_1 and V_2 respectively. Let

$$\Phi_\alpha(k) = T_\alpha(\tau^*_\alpha(k)\xi_\alpha), \qquad k \in K,\ \alpha = 1, 2.$$

Then Φ_1 and Φ_2 are unit vectors in $\mathscr{H}_\sigma$. Define

$$\Phi_{\alpha i}(k) = T_\alpha(\tau^*_\alpha(k)\xi_{\alpha i}), \qquad k \in K,\ \alpha = 1, 2, \text{ and } 1 \leqslant i \leqslant t_\alpha.$$

Then $\{\Phi_{\alpha i}\}$ is an orthonormal basis of V'_α. Also

$$f(x) = (\Phi_1, \pi_{\sigma,\Lambda}(x)\Phi_2) = (\pi_{\sigma,\Lambda}(x^{-1})\Phi_1, \Phi_2).$$

This is equal to the expression

$$\int_K (T_1[\tau^*_1(K(xk))\xi_1], T_2[\tau^*_2(k)\xi_2])e^{(i\Lambda\mu_0 - \rho)(H(xk))}\, dk. \tag{5.2}$$

The inner product in this integrand is on V_σ, the space on which σ acts. $H(xk)$ and $K(xk)$ were defined in §2. Combining the formulae (5.1) and (5.2) we obtain the following formula

$$\begin{aligned} \phi(x) = (t_1t_2)^{-1/2} \sum_{ij} \xi_{1i} \otimes \xi^*_{2j} \cdot \int_K &(T_1[\tau^*_1(K(xk))\xi_{1i}], \\ &T_2[\tau^*_2(k)\xi_{2j}])e^{(i\Lambda\mu_0 - \rho)(H(xk))}\, dk. \end{aligned} \tag{5.3}$$

Let $L = L^\tau$ be the following set of functions on M:

$$\{\psi\colon M \to V_\tau\colon \psi(m_1mm_2) = \tau(m_1)\psi(m)\tau(m_2),\ m, m_1, m_2 \in M\}.$$

L^τ is a Hilbert space with inner product

$$\begin{aligned} (\psi_1, \psi_2)_M &= \int_M (\psi_1(m), \psi_2(m))\, dm \\ &= \int_M (\tau_1(m)\psi_1(1), \tau_1(m)\psi_2(1))\, dm = (\psi_1(1), \psi_2(1)). \end{aligned}$$

If ψ is in L^τ, then $\psi(1)$ is in $V_1 \otimes V_2^*$ and it can be regarded as an intertwining operator from V_2 to V_1 for $\tau_2|_M$ and $\tau_1|_M$. Conversely, if S is such an intertwining operator, then

$$\psi(m) = \tau_1(m)S = S\tau_2(m)$$

is a function in L^τ.

If σ is in $\mathscr{E}_M$, let L^τ_σ be the set of functions ψ in L^τ such that $\psi(m)$ transforms under left and right translates of M according to the representation σ of M. Then there exists a finite number of representations $\{\sigma_1, \ldots, \sigma_r\}$ in $\mathscr{E}_M$ such that

$$L^\tau = L^\tau_{\sigma_1} \oplus \cdots \oplus L^\tau_{\sigma_r}.$$

For any ψ in L^τ let us extend the domain of ψ to all of G by defining

$$\psi(kan) = \tau_1(k)\psi(1), \qquad k \in K,\ a \in A_\mathfrak{p},\ n \in N.$$

Let us return to our function $\phi(x)$ above. Define

$$T_1^*: V_\sigma \to V_1$$

as the adjoint of T_1. Let

$$S = T_1^* T_2: V_2 \to V_1.$$

S is an intertwining operator for $\tau_2|_M$ and $\tau_1|_M$. S is also canonically an element in $V_1 \otimes V_2^*$. Note that

$$S = (t_1 t_2)^{-1/2} \sum_{ij} \xi_{1i} \otimes \xi_{2j}^* (\xi_{1i}, S\xi_{2j})_{V_1}.$$

Therefore

$$S = (t_1 t_2)^{-1/2} \sum_{ij} \xi_{1i} \otimes \xi_{2j}^* (T_1\xi_{1i}, T_2\xi_{2j})_{V_\sigma}. \tag{5.4}$$

The subscripts V_1 and V_σ indicate in what space the inner product is taken. Then the function

$$\psi(m) = \tau_1(m)S = S\tau_2(m), \qquad m \in M,$$

is in L^τ_σ. Also

$$\begin{aligned}\|\psi\|_M^2 &= (\psi(1), \psi(1)) = (S, S) \\ &= (t_1 t_2)^{-1/2} \sum_{ij} |(T_1\xi_{1i}, T_2\xi_{2j})|^2.\end{aligned}$$

Since $\|T_\alpha\| = (t_\alpha)^{1/2}$, it can be shown from (4.5) that this last expression is equal to $\dim \sigma$.

From (5.3) and (5.4) we obtain the formula

$$\phi(x) = \int_K \psi(xk)\tau(k^{-1}) e^{(i\Lambda\mu_0 - \rho)(H(xk))}\, dk. \tag{5.5}$$

For any ψ in L^τ we write

$$E_\Lambda(\psi: x) = \int_K \psi(xk)\tau(k^{-1}) e^{(i\Lambda\mu_0 - \rho)(H(xk))}\, dk.$$

$E_\Lambda(\psi: x)$ is called the Einsenstein integral of ψ and Λ.

Suppose, conversely, that we were given ψ in L^τ_σ such that $\|\psi\|_M^2 = \dim \sigma$. Then we could choose T_α in $R(\tau_\alpha, \sigma)$ with $\|T_\alpha\|^2 = \dim \tau_\alpha$ for $\alpha = 1$ or 2 such that

$$\psi(1) = T_1^* T_2.$$

Again we can define

$$\Phi_{\alpha i}(k) = T_\alpha(\tau_\alpha^*(k)\xi_{\alpha i}), \qquad k \in K, \ \alpha = 1, 2.$$

Then $\Phi_{\alpha i}$ is a unit vector in $\mathscr{H}_\sigma$. Working backward we can obtain the formula

$$E_\Lambda(\psi : x) = (t_1 t_2)^{-1/2} \sum_{ij} \xi_{1i} \otimes \xi_{2j}^*(\Phi_{1i}, \pi_{\sigma,\Lambda}(x)\Phi_{2j}). \tag{5.6}$$

Now, if $\psi(1) = T_1^* T_2$ as above, and $\Lambda \neq 0$, then $n_\sigma(\Lambda)T_\alpha$ is in $R(\tau_\alpha, \sigma')$ and $\|T_\alpha\|^2 = t_\alpha$ for α equal to 1 or 2. Define

$$(M_\sigma(\Lambda)\psi)(1) = (n_\sigma(\Lambda)T_1)^*(n_\sigma(\Lambda)T_2). \tag{5.7}$$

Then $M_\sigma(\Lambda)\psi$ can be regarded as a function in $L_{\sigma'}^\tau$. It has the same norm as ψ. Therefore, $M_\sigma(\Lambda)$ is a unitary map of L_σ^τ onto $L_{\sigma'}^\tau$. We can then define a unitary linear transformation $M(\Lambda)$ of L^τ by defining it to be $M_\sigma(\Lambda)$ on each of the orthogonal subspaces L_σ^τ of L^τ.

If $\Lambda \neq 0$ we have the equation

$$(\Phi_1, \pi_{\sigma,\Lambda}(x)\Phi_2) = (N_\sigma(\Lambda)\Phi_1, \pi_{\sigma',-\Lambda}(x)N_\sigma(\Lambda)\Phi_2).$$

Then from (5.6) we obtain the formula

$$E_\Lambda(\psi : x) = E_{-\Lambda}(M(\Lambda)\psi : x). \tag{5.8}$$

This is the functional equation for the Eisenstein integral.

From (5.7) and (2.14) we obtain the formula

$$M(\Lambda)^{-1} = M(-\Lambda). \tag{5.9}$$

Since $M(\Lambda)$ is unitary, it is clear that

$$M(\Lambda)^* = M(\Lambda)^{-1} = M(-\Lambda). \tag{5.10}$$

We make a final remark about the irreducibility of the representations $\pi_{\sigma,0}$, for σ in $\mathscr{E}_M$. These representations may or may not be irreducible. If $\pi_{\sigma,0}$ is irreducible, then for any nonzero vectors Φ_1 and Φ_2 in $\mathscr{H}_{\sigma,0}$ the function $(\Phi_1, \pi_{\sigma,0}(x)\Phi_2)$ cannot vanish identically in x. Therefore if τ is in $\mathscr{E}_K^2$ and ψ is any nonzero vector in L_σ^τ, $E_0(\psi : x)$ does not vanish identically in x.

On the other hand, suppose that

$$\mathscr{H}_{\sigma,0} = \mathscr{H}_1 \oplus \mathscr{H}_2$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ are nonzero closed subspaces of $\mathscr{H}_{\sigma,0}$ which are invariant under $\pi_{\sigma,0}$. It is possible to choose nonzero vectors Φ_α in $\mathscr{H}_\alpha$ such that Φ_α transforms under $\pi_{\sigma,0}|_K$ according to some irreducible representation τ_α of K, for α equal to 1 or 2. Let τ be the double representation (τ_1, τ_2). Then there exists a nonzero element ψ in L_σ^τ such that $E_0(\psi : x)$ vanishes identically in x.

6. Proof that the map is injective. Let π be a unitary representation of G on a Hilbert space $\mathscr{H}$. If v is a vector in $\mathscr{H}$ such that the map from G to $\mathscr{H}$ given by

$$x \to \pi(x)v, \qquad x \in G,$$

is infinitely differentiable, v is called a differentiable vector. Let $\mathscr{H}^\infty$ be the set of differentiable vectors in $\mathscr{H}$. If v is in $\mathscr{H}^\infty$ and X is in $\mathfrak{g}$, define

$$\pi(X)v = \lim_{t \to 0} \frac{1}{t}(\pi(\exp tX)v - v).$$

It can be checked that this gives a representation of the Lie algebra $\mathfrak{g}$ on the vector space $\mathscr{H}^\infty$. It extends to a representation, again denoted π, of the universal enveloping algebra, $\mathscr{B}$, of $\mathfrak{g}_{\mathbf{C}}$.

Let $\mathfrak{Z}$ be the center of $\mathscr{B}$. If the restriction of π to $\mathfrak{Z}$ is one-dimensional, we obtain a homomorphism

$$\chi \colon \mathfrak{Z} \to \mathbf{C}.$$

In this case we say that π is quasisimple, and we call χ the infinitesimal character of π. It is known that any irreducible unitary representation of G is quasisimple.

Let η be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the real form $\mathfrak{g}$. We define three involutions on $\mathfrak{g}_{\mathbf{C}}$ by

$$\begin{aligned} X^* &= -\eta X, && X \in \mathfrak{g}_{\mathbf{C}}, \\ X^+ &= -X, && X \in \mathfrak{g}_{\mathbf{C}}, \\ \overline{X} &= \eta X, && X \in \mathfrak{g}_{\mathbf{C}}. \end{aligned}$$

If X and Y are in $\mathfrak{g}_{\mathbf{C}}$ and c is a complex number, it is easy to show that

$$\begin{aligned} [\overline{X}, \overline{Y}] &= \overline{[X, Y]}, && \overline{(cX)} = \bar{c}\overline{X}, \\ [X^+, Y^+] &= -[X, Y]^+, && (cX)^+ = cX^+, \\ [X^*, Y^*] &= -[X, Y]^*, && (cX)^* = \bar{c}X^*. \end{aligned}$$

All three involutions extend to involutions of $\mathscr{B}$.

If π is a unitary representation of G, then for g in $\mathscr{B}$,

$$\pi(g^*) = \pi(g)^*$$

where $\pi(g)^*$ is the adjoint operator of $\pi(g)$.

LEMMA 6. *Suppose that π is quasisimple. Assume Φ_1 and Φ_2 are vectors in $\mathscr{H}$ such that the vector spaces*

$$\mathrm{sp}\{\pi(k)\Phi_2 \colon k \in K\}, \qquad \alpha = 1, 2,$$

are both finite-dimensional. Then Φ_1 and Φ_2 are in $\mathscr{H}^\infty$. Furthermore if g_1 and g_2 are in $\mathscr{B}$, and

$$f(x) = (\Phi_1, \pi(x)\Phi_2),$$

then

$$f(g_1; x; g_2) = (\pi(g_1^+)\Phi_1, \pi(x)\pi(\overline{g_2})\Phi_2).$$

PROOF. By [**3(a)**, Theorem 6], Φ_1 and Φ_2 are actually "analytic vectors" for the representation π so in particular they are differentiable. The other statement of the lemma is easy to check. $\square$

Now, in order to discuss the infinitesimal characters of the representations π_ω and $\pi_{\sigma,\Lambda}$ we shall quickly review how Harish-Chandra classifies homomorphisms from $\mathfrak{Z}$ to $\mathbf{C}$.

Let $\mathfrak{M}$, $\mathfrak{A}_\mathfrak{p}$, $\mathfrak{A}_\mathfrak{k}$, and $\mathfrak{A}$ be the universal enveloping algebras of $\mathfrak{m}_\mathbf{c}$, $\mathfrak{a}_{\mathfrak{p},\mathbf{c}}$, $\mathfrak{a}_{\mathfrak{p},\mathbf{c}}$, and $\mathfrak{a}_\mathbf{c}$ respectively. Let $\mathfrak{Z}_M$ be the center of $\mathfrak{M}$. Then $\mathfrak{M}\mathfrak{A}_\mathfrak{p}$ is the universal enveloping algebra of $\mathfrak{m}_\mathbf{c} + \mathfrak{a}_{\mathfrak{p},\mathbf{c}}$, and its center is $\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$.

If z is in $\mathfrak{Z}$, there exists a unique element $\gamma_0'(z)$ in $\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$ such that $z-\gamma_0'(z)$ is in $\sum_{\alpha\in P_+}\mathscr{B}X_\alpha$ [see **3(k)**, Lemma 13].

If z_1 is in $\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$, there exists a unique element $\gamma_1'(z_1)$ in $\mathfrak{A}$ such that $z_1-\gamma_1'(z_1)$ is in $\sum_{\alpha\in P-P_+}\mathfrak{M}\mathfrak{A}_\mathfrak{p}X_\alpha$ [see **3(e)**, Lemma 13].

If z is in $\mathfrak{Z}$, there exists a unique element $\gamma'(z)$ in $\mathfrak{A}$ such that $z-\gamma'(z)$ is in $\sum_{\alpha\in P}\mathscr{B}X_\alpha$ [see **3(e)**, Lemma 18].

Notice that if z is in $\mathfrak{Z}$,

$$z-\gamma_1'(\gamma_0'(z)) = (z-\gamma_0'(z)) + (\gamma_0'(z)-\gamma_1'(\gamma_0'(z))).$$

The right-hand sum is an element in $\sum_{\alpha\in P}\mathscr{B}X_\alpha$. Therefore

$$\gamma_1'\circ\gamma_0' = \gamma'. \tag{6.1}$$

Define automorphisms β and β_1 of $\mathfrak{A}$ by

$$\beta(H) = H+\rho(H), \qquad H\in\mathfrak{a}_\mathbf{c},$$
$$\beta_1(H) = H+\rho_\mathfrak{m}(H), \qquad H\in\mathfrak{a}_\mathbf{c}.$$

Let $\gamma=\beta^{-1}\circ\gamma'$ and let $\gamma_1=\beta_1^{-1}\circ\gamma_1'$. It is known that the maps

$$\gamma\colon\mathfrak{Z}\to\mathfrak{A}, \qquad \gamma_1\colon\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}\to\mathfrak{A},$$

are algebraic isomorphisms onto those elements in $\mathfrak{A}$ which are invariant under W and W_1 respectively [**3(e)**, Lemma 19]. $\mathfrak{A}$ can be regarded as the algebra of polynomial functions from the dual space, $\mathfrak{a}_\mathbf{c}^*$, of $\mathfrak{a}$, into $\mathbf{C}$. If λ is in $\mathfrak{a}_\mathbf{c}^*$, denote the evaluation of p in $S(\mathfrak{a}_\mathbf{c})$ at λ by $\langle p,\lambda\rangle$. Then for any λ in $\mathfrak{a}_\mathbf{c}^*$ define the homomorphism $\chi_\lambda\colon\mathfrak{Z}\to\mathbf{C}$ by

$$\chi_\lambda(z) = \langle\gamma(z),\lambda\rangle, \qquad z\in\mathfrak{Z}.$$

Any homomorphism from $\mathfrak{Z}$ into $\mathbf{C}$ is of this form and $\chi_{\lambda_1}=\chi_{\lambda_2}$ if and only if $\lambda_1=s\lambda_2$ for some s in W. We shall sometimes call χ_λ the homomorphism corresponding to the linear functional λ. Similarly, we can define $\chi_\lambda^M\colon\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}\to\mathbf{C}$ by

$$\chi_\lambda^M(z_1) = \langle\gamma_1(z_1),\lambda\rangle, \qquad z_1\in\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}.$$

Define an automorphism β_0 as follows:

$$\beta_0(X) = X, \qquad X\in\mathfrak{m}_\mathbf{c},$$
$$\beta_0(H) = H+\rho(H), \qquad H\in\mathfrak{a}_\mathfrak{p}.$$

β_0 takes $\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$ onto itself. Put $\gamma_0 = \beta_0^{-1} \circ \gamma_0'$. By (2.5) and (6.1) we see that

$$\gamma = \gamma_1 \circ \gamma_0. \tag{6.2}$$

Now, what are infinitesimal characters of the representations in $\mathscr{E}_d$? It is known that all Cartan subalgebras of the complex Lie algebra $\mathfrak{g}_\mathfrak{c}$ are conjugate under an element in $G_\mathfrak{c}$. Fix y in $G_\mathfrak{c}$ such that $\mathrm{Ad}\, y \cdot \mathfrak{b}_\mathfrak{c} = \mathfrak{a}_\mathfrak{c}$. Then $\mathrm{Ad}\, y$ preserves the Killing form on $\mathfrak{g}_\mathfrak{c}$, and $\mathrm{Ad}\, y$ maps the roots of $(\mathfrak{g}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c})$ onto the roots of $(\mathfrak{g}_\mathfrak{c}, \mathfrak{a}_\mathfrak{c})$. Therefore $\mathrm{Ad}\, y$ maps $(-1)^{1/2}\mathfrak{b}$ onto $(-1)^{1/2}\mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$. We can also assume that $\mathrm{Ad}\, y$ maps the positive roots of $(\mathfrak{g}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c})$ onto the positive roots of $(\mathfrak{g}_\mathfrak{c}, \mathfrak{a}_\mathfrak{c})$. If λ is in L, define

$$\lambda_y(H) = \lambda(\mathrm{Ad}\, y \cdot H), \qquad H \in \mathfrak{a}_\mathfrak{c}.$$

Then

$$B(\lambda, \lambda) = B(\lambda_y, \lambda_y), \qquad \lambda \in L. \tag{6.3}$$

Also

$$\tilde{\omega}^\mathfrak{b}(\lambda) = \tilde{\omega}^\mathfrak{a}(\lambda_y), \qquad \lambda \in L. \tag{6.4}$$

If λ is in L' and $\omega = \omega(\lambda)$ is in $\mathscr{E}_d$, then by [**3(l)**, Theorems 15 and 16], χ_{λ_y} is the infinitesimal character of any representation in the equivalence class of ω. Write L_y and L_y' as the image of L and L' under y.

Now if σ is in $\mathscr{E}_M$, let μ be a real linear functional on $(-1)^{1/2}\mathfrak{a}_\mathfrak{k}$ associated with σ. Regard μ as a linear functional on $(-1)^{1/2}\mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ by making it equal zero on $\mathfrak{a}_\mathfrak{p}$. By looking at a highest weight vector for σ, we can easily check that for any z_1 in $\mathfrak{Z}_M$

$$\sigma(z_1) = \chi_\mu^M(z_1) = \langle \gamma_1(z_1), \mu \rangle. \tag{6.5}$$

LEMMA 7. *Fix σ in $\mathscr{E}_M$ and Λ in* $\mathbf{R}$. *Then for any z in $\mathfrak{Z}$,*

$$\pi_{\sigma,\Lambda}(z) = \chi_{-\mu - i\Lambda\mu_0}(z).$$

PROOF. It is known that the representation $\pi_{\sigma,\Lambda}$ is quasi-simple [**3(a)**, p. 243]. Therefore, there exists a complex linear functional λ on $\mathfrak{a}_\mathfrak{c}$ such that

$$\pi_{\sigma,\Lambda}(z) = \chi_\lambda(z), \qquad z \in \mathfrak{Z}.$$

Our job is to evaluate λ.

Choose a τ in $\mathscr{E}_K^2$ such that $L_\sigma^\tau \neq 0$. Fix ψ in L_σ^τ so that $\|\psi\|_M^2 = \dim \sigma$. Then by (5.6) (using the notation in that formula) and Lemma 6, we obtain the formula

$$\begin{aligned}\overline{\chi_\lambda(\bar{z})} \cdot E_\Lambda(\psi\colon x) &= \sum_{ij} (t_1 t_2)^{-1/2} \cdot \xi_{1i} \otimes \xi_{2j}^* \cdot (\Phi_{1i}, \pi_{\sigma,\Lambda}(x)\pi_{\sigma,\Lambda}(\bar{z})\Phi_{2j}) \\ &= E_\Lambda(\psi\colon x; z), \qquad z \in \mathfrak{Z}.\end{aligned}$$

Let $F(x) = \psi(x)e^{(i\Lambda\mu_0 - \rho)(H(x))}$, and define

$$F(x\colon k) = F(xk)\tau(k^{-1}), \qquad x \in G,\ k \in K.$$

Then by (5.5),

$$E_\Lambda(\psi\colon x) = \int_K F(x\colon k)\,dk.$$

Let z be an arbitrary element in $\mathfrak{Z}$. It can be regarded as a left and right invariant differential operator, so

$$F(x; z\colon k) = F(xk; z)\tau(k^{-1}).$$

Therefore

$$\begin{aligned} E_\Lambda(\psi\colon x; z) &= \int_K F(x; z\colon k)\,dk \\ &= \int_K F(xk; z)\tau(k^{-1})\,dk. \end{aligned}$$

Clearly $F(xn) = F(x)$ for any n in N, so if g is in $\mathscr{B}\mathfrak{n}$, $F(x; g) = 0$. Therefore

$$\begin{aligned} E_\Lambda(\psi\colon x; z) &= \int_K F(xk; \gamma_0'(z))\tau(k^{-1})\,dk \\ &= \int_K F(xk; \beta_0\gamma_0(z))\tau(k^{-1})\,dk. \end{aligned}$$

Suppose that

$$\gamma_0(z) = \sum_i z_i h_i, \qquad z_i \in \mathfrak{Z}_M,\ h_i \in \mathfrak{A}_\mathfrak{p}.$$

Then $\beta_0\gamma_0(z) = \sum_i \beta_0(h_i)$.

Now $F(y \exp tH_0) = F(y)(i\Lambda t - \rho(tH_0))$, so for any h in $\mathfrak{A}_\mathfrak{p}$,

$$\begin{aligned} F(y; \beta_0(h)) &= F(y)\langle \beta_0(h), i\Lambda\mu_0 - \rho\rangle \\ &= F(y)\langle h, i\Lambda\mu_0\rangle. \end{aligned}$$

On the other hand, if m is in M,

$$\begin{aligned} F(ym) &= \psi(ym)e^{(i\Lambda\mu_0-\rho)(H(y))} \\ &= \tau_1(K(y))\cdot T_1^*\cdot\sigma(m)\cdot T_2\cdot e^{(i\Lambda\mu_0-\rho)(H(y))} \end{aligned}$$

where $\psi(1) = T_1^*T_2$ in the notation of §5. Therefore, if z_M is in $\mathfrak{Z}_M$

$$\begin{aligned} F(y; z_M) &= \tau_1(K(y))\cdot T_1^*\cdot\sigma(z_M)\cdot T_2\cdot e^{(i\Lambda\mu_0-\rho)(H(y))} \\ &= \tau_1(K(y))\cdot T_1^*\cdot\langle\gamma_1(z_M),\mu\rangle\cdot T_2\cdot e^{(i\Lambda\mu_0-\rho)(H(y))} \\ &= \langle\gamma_1(z_M),\mu\rangle F(y). \end{aligned}$$

Therefore we see that

$$\begin{aligned} F(y; \beta_0\gamma_0(z)) &= F(y)\left\langle \sum_i \gamma_1(z_i)h_i, \mu + i\Lambda\mu_0\right\rangle \\ &= F(y)\langle\gamma_1\gamma_0(z), \mu + i\Lambda\mu_0\rangle \\ &= F(y)\langle\gamma(z), \mu + i\Lambda\mu_0\rangle. \end{aligned}$$

Therefore,

$$E_\Lambda(\psi\colon x; z) = E_\Lambda(\psi\colon x)\langle\gamma(z), \mu + i\Lambda\mu_0\rangle.$$

It follows that

$$\overline{\chi_\lambda(\bar{z})} = \langle\gamma(z), \mu + i\Lambda\mu_0\rangle.$$

However, z was an arbitrary element in $\mathfrak{Z}$. It is easy to show, then, that $\lambda = -\mu - i\Lambda\mu_0$. The proof of Lemma 7 is complete. $\square$

The linear transformation θ is orthogonal with respect to the bilinear form B. θ has eigenvalues of $+1$ and -1 on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ and $\mathfrak{a}_{\mathfrak{k}}$ respectively. This implies that $\mathfrak{a}_{\mathfrak{p}}$ is orthogonal to $(-1)^{1/2}\mathfrak{a}_{\mathfrak{p}}$ with respect to B.

Suppose the rank of $\mathfrak{g}$ equals n. Then let H_1 and $H_2, \ldots, H_n$ be orthonormal bases of $\mathfrak{a}_{\mathfrak{p}}$ and $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ respectively. For any α in P, fix root vectors X_α and $X_{-\alpha}$ in such a way that $B(X_\alpha, X_{-\alpha}) = 1$. Then $[X_\alpha, X_{-\alpha}] = H_\alpha$. (If λ is any linear functional on $\mathfrak{a}_{\mathfrak{c}}$, define H_λ to satisfy the property

$$B(H_\lambda, H) = \lambda(H), \qquad H \in \mathfrak{a}_{\mathfrak{c}}.)$$

The Casimir element of $\mathfrak{B}$ is given by

$$\omega_{\mathfrak{g}} = H_1^2 + \cdots + H_n^2 + \sum_{\alpha \in P} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha). \tag{6.6}$$

$\omega_{\mathfrak{g}}$ is in $\mathfrak{Z}$ and is equal to the expression

$$H_1^2 + \cdots + H_n^2 + \sum_{\alpha \in P} ([X_\alpha, X_{-\alpha}] + 2X_{-\alpha} X_\alpha),$$

so we see that

$$\gamma'(\omega_{\mathfrak{g}}) = H_1^2 + \cdots + H_n^2 + \sum_{\alpha \in P} H_\alpha = \sum_i H_i^2 + 2H_\rho. \tag{6.7}$$

Therefore,

$$\begin{aligned}
\gamma(\omega_{\mathfrak{g}}) &= \sum_i (H_i - \rho(H_i))^2 + 2H_\rho - 2\rho(H_\rho) \\
&= \sum_i H_i^2 - \sum_i 2\rho(H_i)H_i + \sum_i \rho(H_i) \cdot \rho(H_i) + 2H_\rho - 2B(\rho, \rho) \\
&= \sum_i H_i^2 - B(\rho, \rho).
\end{aligned}$$

Then if λ is any linear functional on $\mathfrak{a}_{\mathfrak{c}}$, we have the formula

$$\chi_\lambda(\omega_{\mathfrak{g}}) = B(\lambda, \lambda) - B(\rho, \rho). \tag{6.8}$$

Suppose that π_ω is a representation in the class of some ω in $\mathcal{E}_d$. Then from (6.3) and (6.8) we obtain the formula

$$\pi_\omega(\omega_{\mathfrak{g}}) = |\omega|^2 - B(\rho, \rho). \tag{6.9}$$

The restriction of any root in P_+ to $\mathfrak{a}_{\mathfrak{p}}$ is equal to either μ_0 or $2\mu_0$. Suppose there are r_1 and r_2 roots in P_+ of each type. Then

$$B(H_0, H_0) = 2 \sum_{\beta \in P_+} \beta(H_0)^2 = 2(r_1 + 4r_2).$$

Let $r^2 = 2(r_1 + 4r_2)$. Then $B(\mu_0, \mu_0) = r^{-2}$. If μ is any linear functional on $\mathfrak{a}_{\mathfrak{k}}$,

$$B(-\mu - i\Lambda\mu_0, -\mu - i\Lambda\mu_0) = B(\mu, \mu) - r^{-2}\Lambda^2.$$

Therefore if σ is in $\mathscr{E}_M$, and Λ is in $\mathbf{R}$,

$$\pi_{\sigma,\Lambda}(\omega_{\mathfrak{g}}) = |\sigma|^2 - r^{-2}\Lambda^2 - B(\rho,\rho). \tag{6.10}$$

The Lie algebra $\mathfrak{k}$ is reductive, so $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2$, where $\mathfrak{k}_1$ is semisimple and $\mathfrak{k}_2$ is abelian. Let $\mathscr{K}$ be the universal enveloping algebra of $\mathfrak{k}_{\mathbf{c}}$, and let $\mathfrak{Z}_K$ be its center. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{k}$ defined at the end of §2. For linear functionals ν on $\mathfrak{h}_{\mathbf{c}}$ we can define the homomorphism

$$\chi_\nu^K : \mathfrak{Z}_K \to \mathbf{C}.$$

Since the Killing form of $\mathfrak{g}_{\mathbf{c}}$ when restricted to $\mathfrak{k}_1$ is K-invariant, this Killing form is a linear combination of the Killing forms of $\mathfrak{k}_{1,i}$ on $\mathfrak{k}_1$, where $\{\mathfrak{k}_{1,i}\}$ are the simple ideals of $\mathfrak{k}_1$. Then it is clear that we can choose an element $\omega_{\mathfrak{k}}$ in $\mathfrak{Z}_K$ such that

$$\chi_\nu^K(\omega_{\mathfrak{k}}) = B(\nu,\nu) - B(\rho_K,\rho_K) \tag{6.11}$$

for any linear functional ν on $\mathfrak{h}_{\mathbf{c}}$.

Notice that

$$\begin{aligned} &\omega_{\mathfrak{g}}^* = \omega_{\mathfrak{g}}, \qquad \overline{\omega}_{\mathfrak{g}} = \omega_{\mathfrak{g}}, \qquad \omega_{\mathfrak{g}}^+ = \omega_{\mathfrak{g}}, \\ &\omega_{\mathfrak{k}}^* = \omega_{\mathfrak{k}}, \qquad \overline{\omega}_{\mathfrak{k}} = \omega_{\mathfrak{k}}, \qquad \omega_{\mathfrak{k}}^+ = \omega_{\mathfrak{k}}. \end{aligned} \tag{6.12}$$

LEMMA 8. *The map $f \to \hat{f}_0$ is a continuous map from $\mathscr{C}(G)$ into $\mathscr{C}_0(\hat{G})$.*

PROOF. Fix polynomials p, q_1, q_2. For f in $C_0^\infty(G)$, we have

$$\|\hat{f}_0\|_{p,q_1,q_2} = \sup_0 |(\Phi_{\tau_1,i_1}, \hat{f}_0(\omega)\Phi_{\tau_2,i_2})| \cdot p(|\omega|) \cdot q_1(|\tau_1|) \cdot q_2(|\tau_2|).$$

By $\sup\limits_0$ we mean the supremum over all ω, (τ_1, i_1), (τ_2, i_2). This is an arbitrary continuous seminorm on $\mathscr{C}_0(\hat{G})$.

Choose integers m, n_1, n_2 such that

$$p(|\omega|) \cdot q_1(|\tau_1|) \cdot q_2(|\tau_2|) \le (1 + |\omega|^2)^m \cdot (1 + |\tau_1|^2)^{n_1} \cdot (1 + |\tau_2|^2)^{n_2}.$$

Define elements g_1 and g_2 in $\mathfrak{B}$ by

$$\begin{aligned} g_1 &= (\omega_{\mathfrak{k}} + B(\rho_K,\rho_K) + 1)^{n_1}, \\ g_2 &= (\omega_{\mathfrak{g}} + B(\rho,\rho) + I)^m \cdot (\omega_{\mathfrak{k}} + B(\rho_K,\rho_K) + 1)^{n_2}. \end{aligned}$$

By (6.12), $g_1^+ = g_1$ and $\bar{g}_2 = g_2$. Since Φ_{τ_1,i_1} and Φ_{τ_2,i_2} transform under π_ω according to τ_1 and τ_2 respectively, we see that

$$\begin{aligned} \pi_\omega(g_1)\Phi_{\tau_1,i_1} &= (1 + |\tau_1|^2)^{n_1}\Phi_{\tau_1,i_1}, \\ \pi_\omega(g_2)\Phi_{\tau_2,i_2} &= (1 + |\tau_2|^2)^{n_2} \cdot (1 + |\omega|^2)^m \Phi_{\tau_2,i_2}. \end{aligned}$$

Therefore,

$$\|\hat{f}_0\|_{p,q_1,q_2} \le \sup_0 |(\pi_\omega(g_1)\Phi_{\tau_1,i_1}, \hat{f}_0(\omega)\pi_\omega(g_2)\Phi_{\tau_2,i_2})|.$$

If $h(x)$ is the function $(\Phi_{\tau_1,i_1}, \pi_\omega(x)\Phi_{\tau_2,i_2})$, we see from Lemma 6 that

$$\begin{aligned}\|\hat{f}_0\|_{p,q_1,q_2} &\leq \sup_0 \left|\int_G \overline{f(x)} \cdot h(g_1; x; g_2)\, dx\right| \\ &= \sup_0 \left|\int_G \overline{f(g_1; x; g_2)} \cdot h(x)\, dx\right| \\ &\leq \sup_0 \left(\int_G |f(g_1; x; g_2)|^2\, dx\right)^{1/2} \cdot \left(\int_G |h(x)|^2\, dx\right)^{1/2}.\end{aligned}$$

By (2.8) this last expression equals

$$\sup_0 \left(\int_G |f(g_1; x; g_2)|^2\, dx\right)^{1/2} \cdot \beta(\omega)^{-1/2}.$$

But $\{\beta(\omega)^{-1/2} : \omega \in \mathscr{E}_d\}$ is bounded by Lemma 3. Also, by (4.2),

$$\left(\int_G |f(g_1; x; g_2)|^2\, dx\right)^{1/2} \leq \sup_{x\in G}(\Xi(x)^{-1} \cdot (1+\sigma(x))^{r_0/2} \cdot |f(g_1; x; g_2)|) \cdot N(r_0)^{1/2}.$$

We have bounded $\|\hat{f}_0\|_{p,q_1,q_2}$ by a continuous seminorm for $\mathscr{C}(G)$ on f. Since $C_0^\infty(G)$ is dense in $\mathscr{C}(G)$, we have proved the lemma. □

LEMMA 9. *Let* $\sup_1$ *denote the supremum over all* (σ, Λ), (τ_1, i_1), (τ_2, i_2). *Then for nonnegative integers* m, m_1, m_2, *and* n, *the continuous seminorms*

$$\|a_1\|_{(m,m_1,m_2,n)} = \sup_1 \left|\left(\frac{d}{d\Lambda}\right)^n [(\Phi_{\tau_1,i_1}, a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}) \cdot (1+|\sigma|^2 - r^{-2}\Lambda^2)^m]\right| \cdot (1+|\tau_1|^2)^{m_1} \cdot (1+|\tau_2|^2)^{m_2}, \qquad a_1 \in \mathscr{C}_1(\hat{G})$$

form a base for the topology of $\mathscr{C}_1(\hat{G})$.

PROOF. By Leibnitz' rule and induction on n, we can see that it is enough to prove the lemma for the seminorms

$$\sup_1 \left|\left(\frac{d}{d\Lambda}\right)^n (\Phi_{\tau_1,i_1}, a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2})\right| \cdot |(1+|\sigma|^2 - r^2\Lambda^2)|^m \cdot (1+|\tau_1|^2)^{m_1} \cdot (1+|\tau_2|^2)^{m_2}, \qquad a_1 \in \mathscr{C}_1(\hat{G}). \tag{6.13}$$

Fix σ in $\mathscr{E}_M$ and τ in $\mathscr{E}_K$. There is a nonzero vector of the form $\Phi_{\tau,i}$ in our basis for $\mathscr{H}_\sigma$ if and only if the representation σ occurs in $\tau|_M$. Suppose that this is the case. Then if τ acts on the finite-dimensional vector space V_τ, σ acts on a subspace of V_τ. The Cartan subalgebras of $\mathfrak{m}$ and $\mathfrak{k}$ are $\mathfrak{a}_\mathfrak{k}$ and $\mathfrak{h}$ respectively, and we have already ordered their dual spaces. Regard σ as a representation $\sigma_0 \times \varepsilon$ of $M^0 \times Z(A)$ as in §2, and let ν and μ be the highest weights of the representations τ and σ_0 respectively. Then

$$|\sigma|^2 = B(\mu+\rho_M, \mu+\rho_M), \qquad |\tau|^2 = B(\nu+\rho_K, \nu+\rho_K).$$

Write $\mathfrak{h} = \mathfrak{a}_{\mathfrak{k}} \oplus \mathfrak{b}_2$ where $\mathfrak{b}_2$ is a Lie algebra of dimension 1 or 0, depending on whether we are in Case I or II. Assume that $B(\mathfrak{a}_{\mathfrak{k}}, \mathfrak{b}_2) = 0$. Extend μ to a linear functional on $\mathfrak{h}$ by letting it equal zero on $\mathfrak{b}_2$.

Let ξ be a highest weight vector in V_τ for σ_0. V_τ is a direct sum of weight spaces for τ. Examine the action of $\tau(\mathfrak{h})$ on ξ. It is clear that there is a linear functional μ_2 on $\mathfrak{h}$ which is zero on $\mathfrak{a}_{\mathfrak{k}}$ and such that $\nu_1 = \mu + \mu_2$ is a weight for τ. Therefore

$$B(\mu, \mu) \leqslant B(\nu_1, \nu_1).$$

However,

$$B(\nu_1, \nu_1)^{1/2} \leqslant B(\nu_1 + \rho_K, \nu_1 + \rho_K)^{1/2} + B(\rho_K, \rho_K)^{1/2}.$$

Since ν is a highest weight for τ, we see by [**5**, Lemma 3, p. 248] that

$$B(\nu_1 + \rho_K, \nu_1 + \rho_K) \leqslant B(\nu + \rho_K, \nu + \rho_K).$$

We have shown that there is a constant C, independent of τ and σ, such that

$$|\sigma| \leqslant |\tau| + C. \tag{6.14}$$

From (6.14) we obtain the additional formula

$$|\Lambda|^2 \leqslant r^2|(1 + |\sigma|^2 - r^2|\Lambda|^2)| + r^2(1 + (C + |\tau|)^2). \tag{6.15}$$

Formulas (6.14) and (6.15) show that any seminorm of the form (4.8) is dominated by a seminorm of the form (6.13). Since the seminorms (4.8) form a base for the topology of $\mathscr{C}_1(\hat{G})$, our lemma is proved. □

LEMMA 10. *The map $f \to \hat{f}_1$ is a continuous map from $\mathscr{C}(G)$ into $\mathscr{C}_1(\hat{G})$.*

PROOF. Let $\|\cdot\|_1$ be an arbitrary continuous seminorm on $\mathscr{C}_1(\hat{G})$. Since $\dim \sigma$, $\dim \tau_1$, and $\dim \tau_2$ are bounded by polynomials in $|\sigma|$, $|\tau_1|$, and $|\tau_2|$ respectively, we can use Lemma 9 to choose integers m, m_1, m_2, n such that for any a_1 in $L_1^2(\hat{G})$,

$$\begin{aligned}\|a_1\|_1 \leqslant \sup_1 &(\dim \sigma \cdot \dim \tau_1 \cdot \dim \tau_2)^{-1/2} \\ &\cdot \left| \left(\frac{d}{d\Lambda}\right)^n [(\Phi_{\tau_1, i_1}, a_1(\sigma, \Lambda)\Phi_{\tau_2, i_2})(|\sigma|^2 - r^{-2}\Lambda^2)^m] \right| \\ &\cdot (1 + |\tau_1|^2)^{m_1} \cdot (1 + |\tau_2|^2)^{m_2}.\end{aligned}$$

Therefore, for any f in $C_0^\infty(G)$

$$\|\hat{f}_1\|_1 \leq \sup_1 (\dim \sigma \cdot \dim \tau_1 \cdot \dim \tau_2)^{1/2} \cdot \left| \left(\frac{d}{d\Lambda}\right)^n \int_G h(g_1; x; g_2) \cdot \overline{f(x)}\, dx \right|$$

by (6.10). Here

$$\begin{aligned} h(x) &= (\Phi_{\tau_1, i_1}, \pi_{\sigma, \Lambda}(x)\Phi_{\tau_2, i_2}), \\ g_1 &= (\omega_{\mathfrak{k}} + B(\rho_K, \rho_K) + 1)^{m_1}, \\ g_2 &= (\omega_{\mathfrak{g}} + B(\rho, \rho) + 1)^m \cdot (\omega_{\mathfrak{k}} + B(\rho_K, \rho_K) + 1)^{m_2}. \end{aligned}$$

Therefore

$$\|\hat{f}_1\|_1 \leqslant \sup_1 (\dim\sigma \cdot \dim\tau_1 \cdot \dim\tau_2)^{-1/2} \cdot \left| \left(\frac{d}{d\Lambda}\right)^n \int_G h(x) \cdot \overline{f(g_1; x; g_2)}\, dx \right| .$$

Let τ be the double representation (τ_1, τ_2) of K. Then by (5.6) there is a ψ in L^τ_σ, with $\|\psi\|^2_M = \dim\sigma$, such that the last expression in the above inequality is bounded by

$$\sup_1 (\dim\sigma)^{-1/2} \cdot \int_G |f(g_1; x; g_2)| \cdot \left| \left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi : x) \right| dx.$$

Now if k is in K,

$$|\psi(k)| = |\psi(1)| = \|\psi\|_M = (\dim\sigma)^{1/2}.$$

Therefore, by (5.5),

$$\begin{aligned}
&(\dim\sigma)^{-1/2} \left| \left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi : x) \right| \\
&\quad = (\dim\sigma)^{-1/2} \left| \int_K \psi(xk)\tau(k^{-1}) \cdot [i\mu_0(H(xk))]^n \cdot e^{(i\Lambda\mu_0 - \rho)(H(xk))}\, dk \right| \\
&\quad \leq \int_K |\mu_0(H(xk))|^n \cdot e^{-\rho(H(xk))}\, dx.
\end{aligned}$$

LEMMA 11. *For any x in G,*

$$|\mu_0(H(x))| \leqslant r^{-1}\sigma(x).$$

Assuming the proof of Lemma 11 for the moment, we see that

$$\begin{aligned}
(\dim\sigma)^{-1/2} \left| \left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi : x) \right| &\leqslant \int_K r^{-n}\sigma(xk)^n e^{-\rho(H(xk))}\, dk \\
&= r^{-n}\sigma(x)^n \cdot \int_K e^{-\rho(H(xk))}\, dk \\
&= r^{-n}\sigma(x)^n \cdot \Xi(x).
\end{aligned}$$

Therefore

$$\|\hat{f}_1\|_1 \leqslant r^{-n} \int_G |f(g_1; x; g_2)| \cdot \sigma(x)^n \cdot \Xi(x)\, dx.$$

By (4.2), the right-hand side of this inequality is bounded by

$$r^{-n} \cdot N(r_0) \cdot \sup_{x \in G} (\Xi(x)^{-1}(1 + \sigma(x))^{n + r_0} \cdot |f(g_1; x; g_2)|).$$

We have dominated $\|\hat{f}_1\|_1$ by a continuous seminorm for $\mathscr{C}(G)$ on f. Since $C_0^\infty(G)$ is dense in $\mathscr{C}(G)$, this is enough to prove Lemma 10. □

We still must prove Lemma 11. If x is in G, we can find k', k in K, and t in $\mathbf{R}$ such that $x = k' \cdot \exp tH_0 \cdot k$. There exists a real number t_k such that

$$H(x) = H(\exp tH_0 \cdot k) = t_k H_0.$$

Now [**3(g)**, Lemma 35] and [**3(g)**, Lemma 35, Corollary 2] establish precisely that $-t \leqslant t_k \leqslant t$. Therefore,

$$|\mu_0(H(x))| = |t_k| \leqslant |t| = r^{-1}\sigma(x). \quad \square$$

Lemmas 8 and 10 show that the Fourier transform is a continuous map from $\mathscr{C}(G)$ into $\mathscr{C}(\hat{G})$. It will be more difficult to prove surjectivity.

7. Theorem 3′ and some elementary formulae. Let $a = (a_0, a_1)$ be an element in $L^2(\hat{G})$. There exists a unique function f in $L^2(G)$ such that $\hat{f} = a$. We wish to find a formula for f. f is the unique function in $L^2(G)$ such that for every g in $C_0^\infty(G)$,

$$\int_G f(x)\overline{g(x)}\,dx = (a, \hat{g}).$$

(The latter inner product is that of $L^2(\hat{G})$.) We shall write $\sum_0$ to denote summation over all ω, (τ_1, i_1), (τ_2, i_2). We write $\sum_1$ to denote summation over all σ, (τ_1, i_1), (τ_2, i_2).

Then $(a, \hat{g})$ equals

$$\begin{aligned}
&\sum_0 (a_0(\omega)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \hat{g}_0(\omega)\Phi_{\tau_2,i_2})\beta(\omega) \\
&\qquad + \sum_1 \int_{-\infty}^{\infty} (a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \hat{g}_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}) \cdot \beta(\sigma,\Lambda)\,d\Lambda \\
&\quad = \sum_0 \int_G (a_0(\omega)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \pi_\omega(x)\Phi_{\tau_2,i_2}) \cdot \beta(\omega)\overline{g(x)}\,dx \\
&\qquad + \sum_1 \int_{-\infty}^{\infty} \int_G (a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \pi_{\sigma,\Lambda}(x)\Phi_{\tau_2,i_2}) \\
&\quad \cdot \beta(\sigma,\Lambda) \cdot \overline{g(x)}\,dx\,d\Lambda.
\end{aligned}$$

Let us assume that the integrals in the last expression are absolutely convergent. This is true for example if a is in $\mathscr{C}(\hat{G})$. Then we may take the integration on G outside. Define functions $\check{a}_0(x)$ and $\check{a}_1(x)$ by

$$\begin{aligned}
\check{a}_0(x) &= \sum_0 (a_0(\omega)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \pi_\omega(x)\Phi_{\tau_2,i_2})\beta(\omega), \\
\check{a}_1(x) &= \sum_1 \int_{-\infty}^{\infty} (a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \pi_{\sigma,\Lambda}(x)\Phi_{\tau_2,i_2})\beta(\sigma,\Lambda)\,d\Lambda.
\end{aligned} \tag{7.1}$$

Then we have the formula

$$\int_G f(x)\overline{g(x)}\,dx = \int_G (\check{a}_0(x) + \check{a}_1(x))\overline{g(x)}\,dx.$$

Since the function $g(x)$ is arbitrary we see that

$$f(x) = \check{a}_0(x) + \check{a}_1(x).$$

By definition of $f(x)$, we have the formulae

$$\hat{\check{a}}_0 = a_0, \qquad \hat{\check{a}}_1 = a_1. \tag{7.2}$$

To prove the surjectivity in Theorem 3, we have to show that both $\check{a}_0(x)$ and $\check{a}_1(x)$ are in $\mathscr{C}(G)$. If h is in $\mathscr{S}(\mathbf{R})$, the Schwartz space of $\mathbf{R}$, we shall be studying the functions

$$f_0(x) = (\Phi_{\tau_1,i_1}, \pi_\omega(x)\Phi_{\tau_2,i_2})\beta(\omega),$$

$$f_1(x) = \int_{-\infty}^{\infty} h(\Lambda)(\Phi_{\tau_1,i_1}, \pi_{\sigma,\Lambda}(x)\Phi_{\tau_2,i_2})\beta(\sigma,\Lambda)\, d\Lambda.$$

If ν is a continuous seminorm on $\mathscr{C}(G)$, we will need to know that both $\nu(f_0)$ and $\nu(f_1)$ are finite. Furthermore, we will have to determine the dependence of $\nu(f_0)$ and $\nu(f_1)$ on the variables τ_1, τ_2, σ, ω, and h.

It follows from the remarks of §5 that the study of the functions $f_0(x)$ and $f_1(x)$ is equivalent to the study of certain (τ_1, τ_2) spherical functions. To complete the proof of Theorem 3 it is enough to prove the following

THEOREM 3$'$. *Let τ be in $\mathscr{E}_K^2$.*

(a) *Let ϕ_ω^τ be a τ-spherical function of the form* (5.1) *corresponding to a representation π_ω with ω in $\mathscr{E}_d$. Then for each g_1, g_2 in $\mathscr{B}$, and real s, there are polynomials p, q such that*

$$\sup_{x\in G} |\phi_\omega^\tau(g_1; x; g_2)\beta(\omega)\Xi(x)^{-1}(1+\sigma(x))^s| \leqslant p(|\omega|)q(|\tau|).$$

(b) *If σ is in $\mathscr{E}_M$, and ψ is in L_σ^τ, with $\|\psi\|_M = 1$, let $E_\Lambda(\psi\colon x)$ be the Eisenstein integral as in* (5.6)*. Then for each g_1, g_2 in $\mathscr{B}$, and real s, there exist polynomials p_1, p_1, q, and an integer N, such that whenever h is in $\mathscr{S}(\mathbf{R})$,*

$$\sup_{x\in G}\left|\int_{-\infty}^{\infty} h(\Lambda)E_\Lambda(\psi\colon g_1; x; g_2)\beta(\sigma,\Lambda)\, d\Lambda\cdot \Xi(x)^{-1}(1+\sigma(x))^s\right|$$
$$\leqslant p_1(|\sigma|)q(|\tau|)\cdot \sup_{1\leqslant t\leqslant N}\cdot \sup_{\Lambda\in\mathbf{R}} (p_2(|\Lambda|))\left|\left(\frac{d}{d\Lambda}\right)^t h(\Lambda)\right|.$$

We shall devote most of our remaining work to proving Theorem 3$'$.

Let $\mathscr{L}_1^2(\hat{G})$ and $\mathscr{S}_1(\hat{G})$ be defined analogously to $L_1^2(\hat{G})$ and $\mathscr{C}_1(\hat{G})$ respectively, but without the symmetry condition

$$a_1(\sigma', -\Lambda) = N_\sigma(\Lambda)a_1(\sigma,\Lambda)N_\sigma(\Lambda)^{-1}.$$

Now suppose that a_1 is in $\mathscr{L}_1^2(\hat{G})$, and that

$$\sum_1 \int_{-\infty}^{\infty} (a_1(\sigma,\Lambda)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})\beta(\sigma,\Lambda)\, d\Lambda$$

is absolutely convergent. This is true in particular if a_1 is in $\mathscr{S}_1(\hat{G})$. Then we define $\check{a}_1(x)$ by the equation in (7.1).

Suppose that for each σ in $\mathscr{E}_M$, $\{\Phi'_{\tau,i}\}$ is another orthonomal basis for $\mathscr{H}_\sigma$ such that each $\Phi'_{\tau,i}$ transforms under π_σ according to the irreducible representation τ of K. Then it is not hard to show that the expression

$$\sum_1 \int_{-\infty}^{\infty} (a_1(\sigma,\Lambda)\Phi'_{\tau_2,i_2}, \Phi'_{\tau_1,i_1})(\Phi'_{\tau_1,i_1}, \pi_{\sigma,\Lambda}(x)\Phi'_{\tau_2,i_2})\beta(\sigma,\Lambda)\, d\Lambda$$

is also absolutely convergent and equal to $\check{a}_1(x)$. Define

$$b_1(\sigma,\Lambda) = \tfrac{1}{2}(a_1(\sigma,\Lambda) + N_\sigma(\Lambda)^{-1}a_1(\sigma',-\Lambda)N_\sigma(\Lambda)).$$

Then, using the above remark we can show that

$$\check{b}_1(x) = \check{a}_1(x).$$

Then by (7.2), $\hat{\check{a}}_1(\sigma,\Lambda) = b(\sigma,\Lambda)$. Therefore, we have the formula

$$\hat{\check{a}}_1(\sigma,\Lambda) = \tfrac{1}{2}[a_1(\sigma,\Lambda) + N_\sigma(\Lambda)^{-1}a_1(\sigma', -\Lambda)N_\sigma(\Lambda)]. \tag{7.3}$$

Now let $\tau = (\tau_1, \tau_2)$ be an irreducible double representation of K. It is of interest to define Fourier transforms for τ-spherical functions on G.

Choose $\sigma_1, \ldots, \sigma_r$ in $\mathscr{E}_M$ such that

$$L^\tau = L^\tau_{\sigma_1} \oplus \cdots \oplus L^\tau_{\sigma_r}.$$

Suppose u is a function which maps $\mathbf{R}$ into L^τ. Then if Λ is in $\mathbf{R}$, write

$$u(\Lambda) = u_1(\Lambda) + \cdots + u_r(\Lambda)$$

where $u_i(\Lambda)$ is the projection of $u(\Lambda)$ onto $L^\tau_{\sigma_i}$. Let us say that u is in $L^2(\mathbf{R};\tau)$ if u_i is in $L^2(\mathbf{R}, \beta(\sigma_i,\Lambda)\,d\Lambda) \otimes L^\tau_{\sigma_i}$ for each i, $1 \leqslant i \leqslant r$. Then if u is in $L^2(\mathbf{R};\tau)$, define

$$\check{u}(x) = \sum_{i=1}^{r} \dim \sigma_i \cdot \lim_{R\to\infty} \int_{-R}^{R} E_\Lambda(u_i(\Lambda) \colon x)\beta(\sigma,\Lambda)\,d\Lambda. \tag{7.4}$$

The limit in this formula is taken in the topology of $L^2(G) \otimes V_\tau$, where V_τ is the space on which τ acts. Then $\check{u}$ is a τ-spherical function which is square-integrable.

On the other hand, if ϕ is a square-integrable τ-spherical function, define a function $\hat{\phi}$ in $L^2(\mathbf{R};\tau)$ as follows. If ψ is an arbitrary element in L^τ then

$$(\hat{\phi}(\Lambda), \psi)_M = \lim_{N\to\infty} \int_{G_N} (\phi(x), E_\Lambda(\psi \colon x))\,dx, \qquad \Lambda \in \mathbf{R}. \tag{7.5}$$

Here G_N is any increasing sequence of compact sets whose union is G, and the limit is taken in the topology of $L^2(\mathbf{R};\tau)$.

Using the formulae (5.6) and (7.3), and recalling the definition of $M(\Lambda)$, it is possible to derive the following formula for any function u in $L^2(\mathbf{R};\tau)$:

$$\hat{\check{u}}(\Lambda) = \tfrac{1}{2}[u(\Lambda) + M(-\Lambda)u(-\Lambda)], \qquad \Lambda \neq 0. \tag{7.6}$$

We shall use this formula in §13.

8. Basic estimates for derivatives. We open this chapter by stating a lemma of E. Nelson.

LEMMA 12. *Let π be an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$. Fix g in $\mathscr{B}$. Then there are an integer $m_g \geqslant 0$ and a constant C, both independent of π, such that for every Φ in $\mathscr{H}^\infty$*

$$\|\pi(g)\Phi\| \leqslant C\|\pi((\omega_{\mathfrak{g}} + \omega_{\mathfrak{k}} - 1)^m)\Phi\|.$$

Here $\omega_{\mathfrak{g}}$ and $\omega_{\mathfrak{k}}$ are the elements in $\mathscr{Z}$ and $\mathscr{Z}_K$ respectively, defined in §6.

For a proof of this lemma see [**7**, Lemma 6.3]. It turns out that m_g can be taken to equal the order of g. ◻

Suppose ρ is the conjugation of $\mathfrak{g}_c$ with respect to the compact real form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. We define a Hermitian scalar product on $\mathfrak{g}_c$ by

$$(X,Y) = -B(X,\rho Y), \qquad X,Y \in \mathfrak{g}_c.$$

We obtain a Hermitian inner product on $\mathfrak{a}_c$ by restriction. This permits us to define an inner product on the dual space of $\mathfrak{a}_c$. If λ is a complex valued linear functional on $\mathfrak{a}_c$, we write $|\lambda|$ as the norm of λ with respect to this inner product.

Suppose τ_1 and τ_2 are in $\mathscr{E}_K$. Write τ as the double representation (τ_1, τ_2) of K. Then

$$|\tau| = |\tau_1| + |\tau_2|, \qquad \dim \tau = \dim \tau_1 \cdot \dim \tau_2.$$

LEMMA 13. *Let π be an irreducible unitary representation of G on the Hilbert space $\mathscr{H}$, with infinitesimal character χ_λ, for λ a linear functional on $\mathfrak{a}_c$. Suppose g_1 and g_2 are in $\mathscr{B}$. Then there exist polynomials p and q, independent of λ, such that the following (somewhat complicated) property is satisfied:*

Whenever Φ_1 and Φ_2 are unit vectors in $\mathscr{H}$ that transform under $\pi|_K$ according to the representations τ_1 and τ_2 in $\mathscr{E}_K$, set $f(x) = (\Phi_1, \pi(x)\Phi_2)$. Then there are two sets, $\{\Psi_{1\alpha}: 1 \leqslant \alpha \leqslant t_1\}$, $\{\Psi_{2\beta}: 1 \leqslant \beta \leqslant t_2\}$, of orthogonal vectors in $\mathscr{H}$, and two sets $\{\tau_{1\alpha}: 1 \leqslant \alpha \leqslant t_1\}$, $\{\tau_{2\beta}: 1 \leqslant \beta \leqslant t_2\}$ of representations in $\mathscr{E}_K$ such that

(i) $(\|\Psi_{1\alpha}\| + \|\Psi_{2\beta}\|) \leqslant p(|\lambda|) \cdot q(|\tau|)$.

(ii) *$\Psi_{1\alpha}$ and $\Psi_{2\beta}$ transform under $\pi|_K$ according to the representations $\tau_{1\alpha}$ and $\tau_{2\beta}$ respectively.*

(iii) $(|\tau_{1\alpha}| + |\tau_{2\beta}|) \leqslant q(|\tau|)$.

(iv) $t_1 + t_2 \leqslant q(|\tau|)$.

(v) $f(g_1; x; g_2) = \sum_{\alpha=1}^{t_1} \sum_{\beta=1}^{t_2} (\Psi_{1\alpha}, \pi(x)\Psi_{2\beta})$.

PROOF. By Lemma 6, $f(x)$ is differentiable. If $\Psi_1 = \pi(g_1^+)\Phi_1$ and $\Psi_2 = (\bar{g}_2)\Phi_2$, we have the formula

$$f(g_1; x; g_2) = (\Psi_1, \pi(x)\Psi_2).$$

By the last lemma, we can choose C and m such that

$$\|\pi(g_1^+)\Phi_1\| \leqslant C\|\pi((\omega_\mathfrak{g} + \omega_\mathfrak{k} - 1)^m)\Phi_1\|.$$

However, Φ_1 transforms under $\pi|_K$ according to τ_1. Therefore, by (6.8) and (6.11) we see that

$$\|\pi((\omega_\mathfrak{g} + \omega_\mathfrak{k} - 1)^m)\Phi_1\| = \|(B(\lambda,\lambda) - B(\rho,\rho) + |\tau_1|^2 - B(\rho,\rho) - 1)^m \Psi_1\|.$$

This last expression is bounded by polynomials in $|\lambda|$ and $|\tau_1|$. Similarly, $\|\Psi_2\|$ is bounded by a product of polynomials in $|\lambda|$ and $|\tau|$.

Suppose that g_1 has order m. Let $\mathscr{B}_m$ be the set of elements in $\mathscr{B}$ of order less than or equal to m. $\mathscr{B}_m$ is a finite-dimensional vector space, and there is a natural representation ν of K on $\mathscr{B}_m$ given by the adjoint map.

Define a linear map from the vector space $\mathscr{B}_m \otimes \mathscr{H}^\infty$ into $\mathscr{H}^\infty$ by

$$g \otimes \Phi \to \pi(g)\Phi, \qquad g \in \mathscr{B}_m, \ \Phi \in \mathscr{H}^\infty.$$

$\mathscr{H}^\infty$ is a K-module under $\pi|_K$, so $\mathscr{B}_m \otimes \mathscr{H}^\infty$ is a K-module. If X is in $\mathfrak{g}$, k is in K and Φ is in $\mathscr{H}^\infty$, then

$$\pi(k)\pi(X)\Phi = \lim_{t\to\infty} \pi(k)\cdot t^{-1}(\pi(\exp tX) - I)\cdot\Phi.$$

This last expression equals

$$\pi(\mathrm{Ad}\, k \cdot X)\cdot \pi(k)\Phi.$$

Therefore, the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{B}_m \otimes \mathscr{H}^\infty & \longrightarrow & \mathscr{H}^\infty \\ \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K} \\ \mathfrak{B}_m \otimes \mathscr{H}^\infty & \longrightarrow & \mathscr{H}^\infty \end{array}$$

Therefore, since Φ transforms under $\pi|_K$ according to τ_1, the vector $\Psi_1 = \pi(g_1^+)\Phi_1$ transforms under $\pi|_K$ as a vector in the space on which $\tau_1 \otimes \nu$ acts. Then

$$\Psi_1 = \Psi_{11} + \cdots + \Psi_{1t_1}, \tag{8.1}$$

where the vectors $\{\Psi_{1\alpha}\}$ are orthogonal to each other, and each $\Psi_{1\alpha}$ transforms under $\pi|_K$ according to some irreducible representation $\pi_{1\alpha}$ of K that occurs in the decomposition of $\tau_1 \otimes \nu$ into irreducible representations of K.

Since (8.1) is an orthogonal decomposition,

$$\|\Psi_{1\alpha}\| \leqslant \|\Psi_1\| \leqslant p(|\lambda|)\cdot q(|\tau|), \qquad 1 \leqslant \alpha \leqslant t_1.$$

Also, we have the formula

$$t_1 \leqslant \dim(\tau_1 \otimes \nu) = \dim \tau_1 \cdot \dim \nu.$$

This expression is bounded by a polynomial in $|\tau_1|$ by the Weyl dimension formula.

Now if $\Psi_{1\alpha}$ is not zero, $\tau_{1\alpha}$ occurs in the decomposition of $\tau_1 \otimes \nu_0$ into irreducible representations, for some irreducible representation ν_0 of K occurring in ν. Let λ_1, λ_0, and $\lambda_{1\alpha}$ be the highest weights of τ_1, ν_0, and $\tau_{1\alpha}$ respectively. By examining the formula for the multiplicity of $\tau_{1\alpha}$ in $\tau_1 \otimes \nu_0$ [**5**, Pg. 262], we see that there is a nonnegative sum, μ, of positive roots, such that

$$(\lambda_1 + \rho_K) + (\lambda_0 + \rho_K) - (\lambda_{1\alpha} + 2\rho_K) = \mu.$$

Therefore

$$(\lambda_1 + \rho_K) + (\lambda_0 + \rho_K) = (\lambda_{1\alpha} + \rho_K) + (\mu + \rho_K).$$

Now $B(\lambda_{1\alpha} + \rho_K, \mu + \rho_K) \geqslant 0$ since $\lambda_{1\alpha} + \rho_K$ is in the positive Weyl chamber. It is then easy to show that

$$|\tau_{1\alpha}| \leqslant |\tau_1| + |\nu_0|$$

if we recall the definitions of $|\tau_{1\alpha}|$, $|\tau_1|$, and $|\nu_0|$. We have now verified conditions (i)–(iv) for $\{\Psi_{1\alpha}\}$ and $\{\tau_{1\alpha}\}$.

We define collections $\{\Psi_{2\beta}\}$ and $\{\tau_{2\beta}\}$ the same way. They satisfy conditions (i)–(iv). It is clear that

$$f(g_1;x;g_2) = \sum_{\alpha=1}^{t_1}\sum_{\beta=1}^{t_2}(\Psi_{1\alpha}, \pi(x)\Psi_{2\beta}).$$

The proof of Lemma 13 is complete. □

Suppose that π equals π_ω, a representation in the equivalence class of some ω in $\mathscr{E}_d$, acting on the Hilbert space $\mathscr{H}_\omega$. Suppose $\omega = \omega(\lambda)$ for some λ in L'. The infinitesimal character of π_ω is χ_{λ_y}, where y is the element in $G_{\mathbf{c}}$ defined in §6. Furthermore, since $\operatorname{Ad} y$ maps $(-1)^{1/2}\mathfrak{b}$ onto $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$, $\rho(\lambda_y)$ equals $-\lambda_y$. Therefore

$$|\lambda_y|^2 = -B(\lambda_y, \rho(\lambda_y)) = B(\lambda_y, \lambda_y) = B(\lambda, \lambda) = |\omega|^2. \tag{8.2}$$

If $\pi = \pi_{\sigma,\Lambda}$, for σ in $\mathscr{E}_M$, choose a real linear functional μ on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ associated with σ. The infinitesimal character of $\pi_{\sigma,\Lambda}$ is $\chi_{-\mu-i\Lambda\mu_0}$, while $\rho(\mu) = -\mu$ and $\rho(i\Lambda\mu_0) = i\Lambda\mu_0$. Therefore

$$\begin{aligned} |-\mu - i\Lambda\mu_0|^2 &= -B(\mu + i\Lambda\mu_0, \rho(\mu + i\Lambda\mu_0)) \\ &= B(\mu,\mu) - B(i\Lambda\mu_0, i\Lambda\mu_0) = |\sigma|^2 + r^{-2}\Lambda^2. \end{aligned} \tag{8.3}$$

9. Preparation for the main estimates. Suppose $\{\phi_\lambda^\tau \colon \lambda \in \mathscr{E}, \tau \in T_\lambda\}$ is a collection of infinitely differentiable τ-spherical functions. τ indexes certain irreducible unitary double representations $\tau = (\tau_1, \tau_2)$ of K on the finite-dimensional Hilbert spaces $V_\tau = V_{\tau_1} \otimes V_{\tau_2}^* \cdot \lambda$ indexes linear functions from $\mathfrak{a}_{\mathbf{c}}$ to $\mathbf{C}$. We have the homomorphisms $\chi_\lambda \colon \mathscr{Z} \to \mathbf{C}$ defined in §6. We assume that

$$z\phi_\lambda^\tau = \chi_\lambda(z)\phi_\lambda^\tau, \qquad z \in \mathfrak{Z}.$$

By $|\lambda|$ we shall mean the real number $[-B(\lambda, \rho(\lambda))]^{1/2}$ as in the last section. We assume that for any g_1, g_2 in $\mathscr{B}$ there are polynomials p and q such that for each x in G

$$|\phi_\lambda^\tau(g_1;x;g_2)| \leqslant p(|\lambda|)q(|\tau|)\Xi(x). \tag{9.1}$$

In [**3(1)**, §27], Harish-Chandra has defined for each ϕ_λ^τ an infinitely differentiable function $\theta = \theta_\lambda^\tau$ mapping $MA_{\mathfrak{p}}$ into V_τ. θ is $\bar{\tau}$-spherical, where $\bar{\tau}$ is the restriction of τ to M. Harish-Chandra shows that θ vanishes if ϕ_λ^τ is in $L^2(G) \otimes V_\tau$. We shall make two assumptions on the collection of linear functionals $\{\lambda \colon \lambda \bullet \mathscr{E}\}$. It then turns out that there exist polynomials p and q, and a number $\varepsilon > 0$, independent of (λ, τ), such that for each $t \geqslant 0$,

$$|e^{\rho(tH_0)}\phi_\lambda^\tau(\exp tH_0) - \theta_\lambda^\tau(\exp tH_0)| \leqslant p(|\lambda|)q(|\tau|)e^{-\varepsilon t}. \tag{9.2}$$

We shall review Harish-Chandra's work and prove the estimate (9.2).

Recall that $\mathfrak{M}\mathfrak{A}_{\mathfrak{p}}$ is the universal enveloping algebra of $\mathfrak{m}_{\mathbf{c}} + \mathfrak{a}_{\mathfrak{p}\mathbf{c}}$; let $\mathfrak{Z}_1 = \mathfrak{Z}_M\mathfrak{A}_{\mathfrak{p}}$ be its center. In §6 we defined isomorphisms

$$\gamma \colon \mathfrak{Z} \to J, \qquad \gamma_1 \colon \mathfrak{Z}_1 \to J_1,$$

where J and J_1 are the elements in $S(\mathfrak{a}_\mathbf{c})$, the symmetric algebra on $\mathfrak{a}_\mathbf{c}$, which are invariant under W and W_1 respectively. (We can regard W_1 as the Weyl group of $\mathfrak{m}_\mathbf{c} + \mathfrak{a}_{\mathfrak{p}\mathbf{c}}$ acting on $\mathfrak{a}_\mathbf{c}$.) J is contained in J_1. We have defined the map

$$\gamma_0 = \gamma_1^{-1} \circ \gamma \colon \mathfrak{Z} \to \mathfrak{Z}_1.$$

Let $\mathfrak{U}_\lambda$ be the annihilator of ϕ_λ^τ in $\mathfrak{Z}$, and let $\mathfrak{U}_{1\lambda} = \mathfrak{Z}_1 \cdot \gamma_0(\mathfrak{U}_\lambda)$. Let $\mathfrak{Z}_1^*$ be the quotient algebra $\mathfrak{Z}_1/\mathfrak{U}_{1\lambda}$. We can regard $\mathfrak{Z}_1^*$ as a complex vector space on which there is a natural $\mathfrak{Z}_1$ action.

If ς is in $\mathfrak{Z}_1$, let ς^* be the projection of ς onto $\mathfrak{Z}_1^*$. Let $\mathfrak{Z}_1^{**}$ be the vector space dual of $\mathfrak{Z}_1^*$. Let $\mathscr{V}_\tau = V_\tau \otimes \mathfrak{Z}_1^{**}$. Make $\mathscr{V}_\tau$ a double K-module by letting K act trivially on $\mathfrak{Z}_1^{**}$. Make it a $\mathfrak{Z}_1$-module by defining

$$\Gamma(z)(v \otimes \varsigma^{**}) = v \otimes z\varsigma^{**}, \qquad z \in \mathfrak{Z}_1,\ v \in V_\tau,\ \varsigma^{**} \in \mathfrak{Z}_1^{**}.$$

(Since $\mathfrak{Z}_1^*$ is a $\mathfrak{Z}_1$-module, there is a natural action of $\mathfrak{Z}_1$ on $\mathfrak{Z}_1^{**}$ obtained by taking transposes.)

Let us examine the algebras J and J_1 more closely, in order to obtain a basis of $\mathfrak{Z}_1^*$. Such results appear in [**3(g)**, §3]. We identify $S = S(\mathfrak{a}_\mathbf{c})$ with the algebra of polynomial functions on $\mathfrak{a}_\mathbf{c}^*$, the dual space of $\mathfrak{a}_\mathbf{c}$. Let $C(S)$, $C(J_1)$, and $C(J)$ be the quotient fields of S, J_1, and J respectively.

LEMMA 14. *If $[W : W_1] = r$, then there are homogeneous elements $v_1 = 1$, $v_2, \ldots, v_r$ in J_1 such that $J_1 = \sum_{1 \leqslant i \leqslant r} Jv_i$. Moreover, the elements $v_1, \ldots, v_r$ are linearly independent over $C(J)$.*

For a proof of this lemma, see [**3(g)**, Lemma 8]. $C(S)/C(J_1)$ and $C(S)/C(J)$ turn out to be normal extensions with Galois groups W_1 and W respectively. □

Now suppose that λ is in $\mathfrak{a}_\mathbf{c}^*$. Let S_λ be the ideal of polynomial functions in S that vanish at λ. Let $J_\lambda = J \cap S_\lambda$ and let $J_{1\lambda} = J_1 \cap S_\lambda$. $J = \mathbf{C} \oplus J_\lambda$ is a vector space decomposition of J and the projection from J onto $J/J_\lambda = \mathbf{C}$ is given by

$$w \to w(\lambda), \qquad w \in J.$$

$J_1 J_\lambda$ is an ideal in J_1 and it is clear that γ_1 defines an isomorphism from $\mathfrak{Z}_1^*$ onto $J_1/J_1 J_\lambda$. We shall obtain a basis of $J_1/J_1 J_\lambda$ over the complex numbers.

We have the formula

$$J_1 = \sum_{1 \leqslant i \leqslant r} Jv_i = \sum_i \mathbf{C}v_i + \sum_i J_\lambda v_i.$$

But $\sum_i J_\lambda v_i$ is in $J_1 J_\lambda$ so $\{v_i : 1 \leqslant i \leqslant r\}$ spans $J_1/J_1 J_\lambda$. On the other hand, suppose $\{c_i\}$ is a set of complex numbers such that the vector $\sum_i c_i v_i$ is in $J_1 J_\lambda$. Now

$$J_1 J_\lambda = \Big(\sum_i Jv_i\Big) J_\lambda = \sum_i J_\lambda v_i.$$

However, $\{v_i : 1 \leqslant i \leqslant r\}$ are linearly independent over $C(J)$ so each c_i is in J_λ. This implies that each c_i equals zero. Therefore, $\{v_i : 1 \leqslant i \leqslant r\}$ is a basis for the vector space $J_1/J_1 J_\lambda$. Let us regard $J_1/J_1 J_\lambda$ as a Hilbert space with orthonormal basis $\{v_1, \ldots, v_r\}$.

Define elements $\eta_1 = 1, \eta_2, \ldots, \eta_r$ in $\mathfrak{Z}_1$ by

$$\gamma_1(\eta_i) = v_i, \qquad 1 \leqslant i \leqslant r.$$

Then $\{\eta_1^*, \ldots, \eta_r^*\}$ is a basis for $\mathfrak{Z}_1^*$. Let $\{\eta_1^{**}, \ldots, \eta_r^{**}\}$ be the dual basis $\mathfrak{Z}_1^{**}$. If we make $\mathfrak{Z}_1^{**}$ into a Hilbert space with orthonormal basis $\{\eta_1^{**}, \ldots, \eta_r^{**}\}$, we can regard $\mathscr{V}_\tau$ as a Hilbert space.

Define a function on $MA_\mathfrak{p}$ by

$$d(ma) = e^{\rho(\log a)}, \qquad a \in A_\mathfrak{p},\ m \in M.$$

We define an automorphism $\varsigma \to \varsigma'$ of $\mathfrak{Z}_1$ by

$$\varsigma' = d^{-1}\varsigma \circ d, \qquad \varsigma \in \mathfrak{Z}_1.$$

(We are regarding $\mathfrak{Z}_1$ as an algebra of differential operators on $M^0A_\mathfrak{p}$.)

If f is in $C^\infty(MA_\mathfrak{p})$, v is in $\mathfrak{M}\mathfrak{A}_\mathfrak{p}$, m is in M, and a is in $A_\mathfrak{p}$, we can define $f(ma;v)$ even though $MA_\mathfrak{p}$ may not be connected. If X is in $\mathfrak{m} + \mathfrak{a}_\mathfrak{p}$, we just write

$$f(ma;X) = \frac{d}{dt} f(ma \exp tX)|_{t=0}.$$

We extend the definition to all v in the universal enveloping algebra $\mathfrak{M}\mathfrak{A}_\mathfrak{p}$ in the usual way. Since $MA_\mathfrak{p} = Z(A)M^0A_\mathfrak{p}$, and since $Z(A)$ and $M^0A_\mathfrak{p}$ commute, any ς in $\mathfrak{Z}_1$ can be regarded as a left and right invariant differential operator on $C^\infty(MA_\mathfrak{p})$.

For m in M and a in $A_\mathfrak{p}$, define

$$\phi_i(ma) = d(ma)\phi_\lambda^\tau(ma;\eta_i').$$

Define

$$\Phi(ma) = \sum_{1 \leqslant i \leqslant r} \phi_i(ma) \otimes \eta_i^{**}.$$

If ς is in $\mathfrak{Z}_1$, choose complex numbers $\{c_{ij} : 1 \leqslant i \leqslant r\}$ such that the differential operators

$$\varsigma\eta_i - \sum_j c_{ij}\eta_j = u_i(\varsigma), \qquad 1 \leqslant i \leqslant r,$$

are in $\mathfrak{U}_{1\lambda}$. Then define

$$\Psi_\varsigma(ma) = \sum_{1 \leqslant i \leqslant r} d(ma)\phi_\lambda^\tau(ma;u_i(\varsigma)') \otimes \eta_i^{**}.$$

Φ and Ψ_ς are functions from $MA_\mathfrak{p}$ to the vector space $\mathscr{V}_\tau$. Also, Φ and Ψ_ς are both $\bar{\tau}$-spherical functions on $MA_\mathfrak{p}$, since elements in $\mathfrak{Z}_1$ act on $C^\infty(MA_\mathfrak{p})$ as left and right invariant differential operators.

LEMMA 15. *Let ς be in $\mathfrak{Z}_1$. Then for m in M and a in $A_\mathfrak{p}$,*

$$\Phi(ma;\varsigma) = \Gamma(\varsigma)\Phi(ma) + \Psi_\varsigma(ma).$$

PROOF. We have the equation

$$\Phi(ma;\varsigma) = \sum_i d(ma)\phi_\lambda^\tau(ma;\varsigma'\eta_i') \otimes \eta_i^{**}.$$

Also,

$$\varsigma'\eta_i' = (\varsigma\eta_i)' = \sum_j c_{ij}\eta_j' + u_i(\varsigma)'.$$

Therefore, $\Phi(ma;\varsigma)$ is equal to

$$\sum_i\sum_j d(ma)\phi_\lambda^\tau(ma;\eta_j') \otimes \eta_i^{**} + \sum_i d(ma)\phi_\lambda^\tau(ma;u_i(\varsigma)'),$$

which in turn equals

$$\sum_{ij} c_{ij}\phi_j(ma) \otimes \eta_i^{**} + \Psi_\varsigma(ma).$$

Since $u_i(\varsigma)^* = 0$, we have the formula

$$(\varsigma\eta_i)^* = \sum_j c_{ij}\eta_j^*.$$

But $\{\eta_i^{**}\}$ is a dual basis of $\{\eta_i^*\}$, so the matrix of the linear transformation ς acting on $\mathfrak{Z}_1^{**}$ is the transpose matrix of its action on $\mathfrak{Z}_1^*$, with respect to these bases. Therefore,

$$\Gamma(H_0)\Phi(ma) = \sum_{ij} c_{ji}\phi_i(ma) \otimes \eta_j^{**} = \sum_{ij} c_{ij}\phi_j(ma) \otimes \eta_i^{**}.$$

This proves the lemma. □

Clearly H_0 is in $\mathfrak{Z}_1$. Write Ψ_{H_0} as Ψ.

COROLLARY. *For any m in M, a in $A_\mathfrak{p}$, and T in $\mathbf{R}$, we have the integral equation*

$$\Phi(ma \cdot \exp TH_0) = e^{T\Gamma(H_0)}\Phi(ma) + \int_0^T e^{(T-t)\Gamma(H_0)}\Psi(ma \cdot \exp tH_0)\,dt.$$

PROOF. If t is in $\mathbf{R}$, we see that

$$\begin{aligned}&\frac{d}{dt}(e^{-t\Gamma(H_0)}\Phi(ma \cdot \exp tH_0))\\ &\quad = e^{-t\Gamma(H_0)}\Phi(ma \cdot \exp tH_0; H_0) - e^{-t\Gamma(H_0)}\Gamma(H_0)\Phi(ma \cdot \exp tH_0)\\ &\quad = e^{-t\Gamma(H_0)}\Psi(ma \cdot \exp tH_0).\end{aligned}$$

Integrate the above equation by $\int_0^T dt$. We obtain the formula

$$e^{-T\Gamma(H_0)}\Phi(ma \exp TH_0) - \Phi(ma) = \int_0^T e^{-t\Gamma(H_0)}\Psi(ma \cdot \exp tH_0)\,dt.$$

Multiply by $e^{T\Gamma(H_0)}$. This proves the corollary. □

Now, fix ς in $\mathfrak{Z}_1$. We would like to obtain an estimate for $\Psi_\varsigma(ma)$. To do this, we must first express the differential operators $u_i(\varsigma)$ in another form.

By Lemma 14, there exist elements w_{ij} in J such that for $1 \leqslant i \leqslant r$,

$$\gamma_1(\varsigma)v_i = \sum_j w_{ij}v_j.$$

Now, the coordinates of $\gamma_1(\varsigma)v_i$ relative to the basis $\{v_j\}$ of J_1/J_1J_λ are clearly $\{w_{ij}(\lambda) : 1 \leqslant j \leqslant r\}$. Then the element

$$\gamma_1(u_i(\varsigma)) = \gamma_1(\varsigma)v_i - \sum_j w_{ij}(\lambda)v_j$$

is in J_1J_λ. $\gamma_1(u_i(\varsigma))$ is equal to

$$\sum_j (w_{ij} - w_{ij}(\lambda))v_j.$$

For each i and j, $(w_{ij} - w_{ij}(\lambda))$ is in J_λ. Let z_{ij}^λ be the differential operator

$$\gamma^{-1}(w_{ij} - w_{ij}(\lambda)).$$

Then

$$u_i(\varsigma) = \sum_j \gamma_0(z_{ij}^\lambda)\gamma_1^{-1}(v_j) = \sum_j \gamma_0(z_{ij}^\lambda)\eta_j.$$

Let u_{ij}^λ be the differential operator

$$z_{ij}^\lambda - \gamma_0(z_{ij}^\lambda)'.$$

u_{ij} is an element of $\mathfrak{B}$, and it is independent of λ. Recall that θ is the Cartan involution of $\mathfrak{g}_c$. In the appendix of [**3(1)**] it is shown that there exist elements N_{ij} in $\mathfrak{n}_c$ and g_{ij} in $\mathscr{B}$, both independent of λ, such that

$$u_{ij} = \theta(N_{ij})g_{ij}, \qquad 1 \leqslant i,\ j \leqslant r.$$

LEMMA 16. *For fixed ς in $\mathfrak{Z}_1$ and h in $\mathfrak{A}_\mathfrak{p}$, there are polynomials p and q and an integer d, independent of (λ, τ), such that for every $t \geqslant 0$*

$$|\Psi_\varsigma(\exp tH_0; h)| \leqslant p(|\lambda|)q(|\tau|)e^{-t}(1+t)^d.$$

PROOF. We have the equations

$$\begin{aligned}\Psi_\varsigma(\exp tH_0; h) &= \sum_i e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0; h'u_i(\varsigma)') \otimes \eta_i^{**} \\ &= \sum_{ij} e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0; h'\eta_j'\gamma_0(z_{ij}^\lambda)') \otimes \eta_i^{**}.\end{aligned}$$

Since z_{ij}^λ is in $\mathfrak{U}_\lambda$, the annihilator of ϕ_λ^τ, this last expression is equal to

$$\begin{aligned}&\sum_{ij} e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0; h'\eta_j'u_{ij}) \otimes \eta_i^{**} \\ &\quad = \sum_{ij} e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0; u_{ij}h'\eta_j') \otimes \eta_i^{**} \\ &\quad = \sum_{ij} e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0; \theta(N_{ij})g_{ij}\eta_j'h') \otimes \eta_i^{**}.\end{aligned}$$

If N is any vector in $\mathfrak{n}_c$, N is equal to $\sum_{\alpha\in P_+} X_\alpha$, where $\{X_\alpha\}$ are vectors in $\mathfrak{g}_c$ such that $[H_0, X_\alpha] = \alpha(H_0)X_\alpha$. But if α is in P_+,

$$\phi_\lambda^\tau(\exp tH_0; \theta(X_\alpha)g_{ij}\eta_j'h') = e^{-t\alpha(H_0)}\phi_\lambda^\tau(\theta(X_\alpha); \exp tH_0; g_{ij}\eta_j'h').$$

But $e^{-t\alpha(H_0)}$ equals either e^{-t} or e^{-2t}. The lemma then follows from (9.1) and (4.1). $\square$

Recall that $\tilde{\omega} = \prod_{\alpha\in P} H_\alpha$. In particular, $\tilde{\omega}$ is in S. For the rest of this section we make the following assumption.

Assumption 1. For each λ in $\mathscr{E}$, $\tilde{\omega}(\lambda)$ is not equal to zero.

We would like to find the eigenvalues of the linear transformation H_0 acting on the vector space J_1/J_1J_λ. We also want to find the norms of the projections of J_1/J_1J_λ onto these eigenspaces. (These are norms as operators on the Hilbert space J_1/J_1J_λ; these projections are not necessarily selfadjoint.)

The field $C(J_1)$ is an extension of degree r of the field $C(J)$. Therefore the trace, $\mathrm{tr}_{C(J_1)/C(J)}$, is a function from $C(J_1)$ into $C(J)$. Define an element v^i in $C(J_1)$ by

$$\mathrm{tr}_{C(J_1)/C(J)}(v^i v_j) = \delta^i_j, \qquad 1 \leqslant i,\ j \leqslant r.$$

($\delta^i_j = 0$ if $i \neq j$, $\delta^i_i = 1$.) Recall that $\tilde{\omega}^{\mathrm{m}} = \prod_{\alpha\in P_M} H_\alpha$. Define

$$D = \tilde{\omega}/\tilde{\omega}^{\mathrm{m}}, \qquad \tau^i = Dv^i.$$

Then D and τ^i are both in $C(S)$. In [**3(g)**, Lemma 12] Harish-Chandra shows that τ^i is actually in S.

Now let $\{s_1 = 1, s_2, \ldots, s_r\}$ be a set of representatives of right cosets of W_1 in W. If v is in $C(J_1)$, then

$$\mathrm{tr}_{C(J_1)/C(J)}(v) = \sum_{1\leqslant i\leqslant r} s_i^{-1}(v).$$

However,

$$\sum_{1\leqslant k\leqslant r} \tau^i(s_k\lambda)\cdot v_j(s_k\lambda)/D(s_k\lambda) = \delta^i_j.$$

Define $A_{ik} = \tau^i(s_k\lambda)$ and $B_{kj} = v_j(s_k\lambda)/D(s_k\lambda)$. Then $A = (A_{ik})$ and $B = (B_{kj})$ are $r\times r$ matrices, and $AB = I$. Define

$$f_{s_i\lambda} = \sum_k \tau^k(s_i\lambda)v_k. \tag{9.3}$$

Then $\{f_{s_i\lambda}\colon 1 \leqslant i \leqslant r\}$ is a base for J_1/J_1J_λ. Also

$$v_j = \sum_k (v_j(s_k\lambda)/D(s_k\lambda))f_{s_k\lambda}. \tag{9.4}$$

LEMMA 17. *If p is in J_1*

$$pf_{s_i\lambda} \equiv p(s_i\lambda)f_{s_i\lambda} \pmod{J_1J_\lambda}.$$

The proof of this lemma is in [**3(g)**, Lemma 15]. $\blacksquare$

In particular, the operator p on J_1/J_1J_λ is semisimple. Since H_0 is in J_1, the lemma tells us that the set of eigenvalues of H_0 is

$$\{\lambda(H_0), \lambda(s_2^{-1}H_0), \ldots, \lambda(s_r^{-1}H_0)\}. \tag{9.5}$$

We can lift J_1 to $\mathfrak{Z}_1$ by γ_1^{-1}. Then if ς is in $\mathfrak{Z}_1$, we have an analogous statement to Lemma 17 for the eigenvalues of the operator $\Gamma(\varsigma)$ on $\mathscr{V}_\tau$. In particular, the eigenvalues of $\Gamma(H_0)$ are also given by (9.5).

Let $\{E'_{s_1}, \dots, E'_{s_r}\}$ be the projections in J_1/J_1J_λ relative to the direct decomposition

$$J_1/J_1J_\lambda = \mathbf{C}f_{s_1\lambda} \oplus \cdots \oplus \mathbf{C}f_{s_r\lambda}.$$

Define

$$E_{s_l} = (\mathrm{Id}) \otimes (\gamma_1^{-1}E'_{s_l}\gamma_1)^*, \qquad 1 \leqslant l \leqslant r,$$

where Id stands for the identity operator on V_τ, and the star denotes the vector space transpose operator. Then the operators $\{E_{s_l}\}$ are the projections of $\mathscr{V}_\tau$ onto the eigenspaces of $\Gamma(\mathfrak{Z}_1)$.

If u is a vector in $\mathscr{V}_\tau$, there are elements $u_1, \dots, u_r$ in V_τ such that

$$u = (u_1 \otimes \eta_1^{**}) + \cdots + (u_r \otimes \eta_r^{**}).$$

Denote u_i by $t_i(u)$.

LEMMA 18. *There is a fixed set of elements $\{p_{ij}^l : 1 \leqslant i,j,l \leqslant r\}$ in S, independent of τ and λ, such that for any u in $\mathscr{V}_\tau$*

$$t_i(E_{s_l}u) = \sum_{j=1}^{r}(p_{ij}^l(\lambda)/\tilde{\omega}(\lambda))t_j(u).$$

PROOF. (9.4) implies the formula

$$E'_{s_l}v_i = E'_{s_l}\left(\sum_{j=1}^{r}(v_i(s_j\lambda)/D(s_j\lambda))f_{s_j\lambda}\right) = (v_i(s_l\lambda)/D(s_l\lambda)) \cdot f_{s_l\lambda}.$$

By (9.3) this last expression is equal to

$$(v_i(s_l\lambda)/D(s_l\lambda)) \cdot \sum_j \tau^j(s_l\lambda)v_j.$$

Now $D(s_l\lambda)$ equals $(\tilde{\omega}(\lambda)/\tilde{\omega}^{\mathrm{m}}(\lambda))\varepsilon$, where ε equals either 1 or -1. This is enough to prove the lemma. □

The set of eigenvalues of either the linear transformation H_0 on J_1/J_1J_λ or $\Gamma(H_0)$ on $\mathscr{V}_\tau$ is given by (9.5). Let Q^+, Q^-, Q^0 be the subsets of these eigenvalues with real parts greater than, less than, or equal to zero, respectively. Let E'^+, E'^-, E'^0, and E^+, E^-, E^0 be the corresponding projections in J_1/J_1J_λ and $\mathscr{V}_\tau$ respectively.

Let ε' be the minimum of the absolute values of the real parts of the numbers in the set $Q^+ \cup Q^-$. (Set $\varepsilon' = 1$ if $Q^+ \cup Q^-$ is empty.)

For the remainder of this chapter we make the following assumption.

Assumption 2. The real parts of all eigenvalues (9.5), as λ ranges in $\mathscr{E}$, generate a lattice in $\mathbf{R}$. In particular, ε' is bounded away from zero independently of λ in $\mathscr{E}$.

LEMMA 19. *The norm of H_0 as an operator on J_1/J_1J_λ is bounded by a polynomial in $|\lambda|$.*

PROOF. $\{v_1, \dots, v_r\}$ is an orthonormal base for J_1/J_1J_λ. By Lemma 14 there exist elements w_{ij} in J, independent of λ, such that

$$H_0v_i = w_{i1}v_1 + \cdots + w_{ir}v_r, \qquad 1 \leqslant i \leqslant r.$$

This means that

$$H_0 v_i \equiv w_{i1}(\lambda)v_1 + \cdots + w_{ir}(\lambda)v_r \pmod{J_1 J_\lambda}.$$

Now $w_{ij}(\lambda)$ is a polynomial function in λ, so it is bounded by a polynomial in $|\lambda|$. Since $\{v_i\}$ is an orthonormal basis for J_1/J_1J_λ, the lemma follows. □

LEMMA 20. *Choose a complex number ξ in the resolvent set of the operator H_0. Let d be the distance from ξ to the spectrum of H_0. Then there are polynomials p_1 and p_2, independent of ξ and λ, such that the norm of the operator $(\xi - H_0)^{-1}$ is bounded by $d^{-r}p_1(|\xi|)p_2(|\lambda|)$.*

PROOF. By Lemma 19 the matrix $W(\lambda) = \{w_{ij}(\lambda)\}$ of the linear transformation H_0 with respect to the orthonormal basis $\{v_1, \ldots, v_r\}$ of J_1/J_1J_λ has entries which are bounded by a polynomial in $|\lambda|$. Let $u_i = (\xi - H_0)v_i$. Then for $1 \leqslant i \leqslant r$

$$u_i = \xi v_i - \sum_j w_{ij}(\lambda)v_j. \tag{9.6}$$

We can solve the equations (9.6) to obtain the formula

$$v_i = \det(\xi I - W(\lambda))^{-1} \cdot \sum_j p_{ij}(\xi, \lambda)u_j$$

where $p_{ij}(\xi, \lambda)$ are polynomial functions of ξ and λ. In particular, each $p_{ij}(\xi, \lambda)$ is bounded by a polynomial in $|\xi|$ and $|\lambda|$. Now

$$\det(\xi I - W(\lambda)) = \prod_{j=1}^{r}(\xi - \lambda(s_j^{-1}H_0)).$$

Therefore

$$|\det(\xi I - W(\lambda))|^{-1} \leqslant d^{-r}.$$

Since $(\xi - H_0)^{-1}u_j = v_j$, our lemma is proved. □

Let ε be the minimum of $\varepsilon'/3$ and $1/3$.

LEMMA 21. *There exists a polynomial p such that*

$$|e^{-t\Gamma(H_0)}E^+| + |e^{t\Gamma(H_0)}E^-| \leqslant p(|\lambda|)e^{-2\varepsilon t}, \qquad t \geqslant 0,$$
$$|e^{t\Gamma(H_0)}E^0| \leqslant p(|\lambda|)e^{\varepsilon t}, \qquad t \in \mathbf{R}.$$

PROOF. It is clearly enough to prove the same statements for the linear transformations $e^{-tH_0}E'^+$, $e^{tH_0}E'^-$, and $e^{tH_0}E'^0$ on the Hilbert space J_1/J_1J_λ.

Let Γ^+, Γ^-, and Γ^0 be closed curves in the complex plane that wind around the corresponding sets of eigenvalues Q^+, Q^-, and Q^0 in a positive sense, but which contain no other eigenvalues. By looking at Assumption 2 and the eigenvalues (9.5) we see that the curves can be chosen to satisfy the following conditions.

(i) $|\xi|$ is bounded by a polynomial in $|\lambda|$ for any ξ on one of the curves.

(ii) The arc length of each of the curves is bounded by a polynomial in $|\lambda|$.

(iii) If ξ is on one of the curves, the distance from ξ to the spectrum of H_0 is not less than ε.

(iv) If ξ is a complex number, let $\xi_{\mathbf{R}}$ be its real part. Then

$$\begin{aligned} \xi_{\mathbf{R}} &\geqslant 2\varepsilon \quad \text{for } \xi \text{ on } \Gamma^+, \\ \xi_{\mathbf{R}} &\leqslant -2\varepsilon \quad \text{for } \xi \text{ on } \Gamma^-, \\ |\xi_{\mathbf{R}}| &\leqslant \varepsilon \quad \text{for } \xi \text{ on } \Gamma^0. \end{aligned}$$

From the spectral theory for a linear transformation on a finite-dimensional vector space we have the formulae

$$\begin{aligned} e^{-tH_0}E'^{+} &= \int_{\Gamma^+} e^{-t\xi}(\xi - H_0)^{-1}\, d\xi, \qquad t \geqslant 0, \\ e^{tH_0}E'^{-} &= \int_{\Gamma^-} e^{t\xi}(\xi - H_0)^{-1}\, d\xi, \qquad t \geqslant 0, \\ e^{tH_0}E'^{0} &= \int_{\Gamma^0} e^{t\xi}(\xi - H_0)^{-1}\, d\xi, \qquad t \in \mathbf{R}. \end{aligned}$$

Therefore if $t \geqslant 0$

$$|e^{-tH_0}E'^{+}| \leqslant \int_{\Gamma^+} |e^{-t\xi}| \cdot |(\xi - H_0)^{-1}|\, d\xi.$$

By Lemma 20 and conditions (iii) and (iv), this last expression is bounded by

$$e^{-2\varepsilon t}e^{-r}p_2(|\lambda|)\int_{\Gamma^+} p_1(|\xi|)\, d\xi.$$

Therefore by conditions (i) and (ii) there exists a polynomial p such that

$$|e^{-tH_0}E'^{+}| \leqslant e^{-2\varepsilon t} \cdot p(|\lambda|), \qquad t \geqslant 0.$$

The inequalities for $|e^{tH_0}E'^{-}|$ and $|e^{tH_0}E'^{0}|$ follow in the same way. □

Write $E^{\pm}\Phi$ as $\Phi^{\pm}$ and $E^0\Phi$ as Φ^0.

LEMMA 22. *For any fixed h in $\mathfrak{A}_p$ there are polynomials p and q such that for any $T \geqslant 0$*

$$|\Phi^-(\exp TH_0; h)| \leqslant p(|\lambda|)q(|\tau|)e^{-\varepsilon t}.$$

PROOF. $\Gamma(H_0)$ and E^- commute. Therefore by the corollary to Lemma 15, $\Phi^-(\exp TH_0; h)$ is equal to

$$e^{T\Gamma(H_0)}\Phi^-(1; h) + \int_0^T e^{(T-t)\Gamma(H_0)} \cdot E^-\Psi(\exp TH_0; h)\, dt.$$

The first term of this expression is easily handled with the help of (9.1) and Lemma 21. On the other hand, Lemma 21 tells us that

$$\begin{aligned} &\left| \int_0^T e^{(T-t)\Gamma(H_0)}E^-\Psi(\exp tH_0; h)\, dt \right| \\ &\quad \leqslant p(|\lambda|)\int_0^T e^{-2\varepsilon(T-t)}|\Psi(\exp tH_0; h)|\, dt \\ &\quad \leqslant p(|\lambda|)e^{-\varepsilon t}\int_0^{T/2} |\Psi(\exp tH_0; h)|\, dt + p(|\lambda|)\int_{T/2}^T |\Psi(\exp tH_0; h)|\, dt. \end{aligned}$$

Our lemma then follows from Lemma 16. □

LEMMA 23. *For any fixed h in $\mathfrak{A}_\mathfrak{p}$ there are polynomials p and q such that for any $T \geqslant 0$*

$$|\Phi^+(\exp TH_0; h)| \leqslant p(|\lambda|)q(|\tau|)e^{-\varepsilon T}.$$

PROOF. By means of a change of variables we can rewrite the integral equation of the corollary of Lemma 15. Then for a in $\mathfrak{A}_\mathfrak{p}$ and t in $\mathbf{R}$

$$\Phi(a; h) = e^{-T\Gamma(H_0)}\Phi(a \cdot \exp TH_0; h) - \int_0^T e^{-t\Gamma(H_0)}\Psi(a \exp tH_0; h)\, dt.$$

Operate on both sides of this equation by E^+, and let T approach ∞. Now $|e^{-T\Gamma(H_0)}E^+|$ decreases exponentially in T. However, by (9.1) and (4.1), $|\Phi(a \exp TH_0; h)|$ is bounded by a polynomial in T. Therefore, the first term of right-hand side of the above equation approaches zero. We have the formula

$$\Phi^+(a; h) = -\int_0^\infty e^{-t\Gamma(H_0)}E^+\Psi(a \exp tH_0; h)\, dt.$$

Let a equal $\exp TH_0$. We obtain the equation

$$\Phi^+(\exp TH_0; h) = -\int_T^\infty e^{-(t-T)\Gamma(H_0)}E^+\Psi(\exp tH_0; h)\, dt.$$

Now $|e^{-(t-T)\Gamma(H_0)}E^+|$ is bounded by a polynomial in $|\lambda|$ if $t \geqslant T$. Our lemma then follows from Lemma 16. □

For a in $A_\mathfrak{p}$, m in M, define

$$\Theta(ma) = \Phi^0(ma) + \int_0^\infty e^{-t\Gamma(H_0)}E^0\Psi(ma \exp tH_0)\, dt.$$

(The integral converges absolutely by Lemmas 16 and 21.) It is clear that for any h in $\mathfrak{A}_\mathfrak{p}$

$$\Theta(ma; h) = \Phi^0(ma; h) + \int_0^\infty e^{-t\Gamma(H_0)}E^0\Psi(ma \exp tH_0; h)\, dt.$$

From the corollary of Lemma 15 we obtain the formula

$$\Theta(ma) = \lim_{T\to\infty} e^{-T\Gamma(H_0)}\Phi^0(ma \exp TH_0).$$

Therefore

$$\Theta(ma \exp tH_0) = e^{t\Gamma(H_0)}\Theta(ma), \qquad m \in M,\ a \in A_\mathfrak{p},\ t \in \mathbf{R}. \tag{9.7}$$

Since Ψ is $\overline{\tau}$-spherical, and since both the left and right actions of $\overline{\tau}(m)$ on $\mathscr{V}_\tau$ commute with E^0, we have the formula

$$\Theta(m_1 mam_2) = \overline{\tau}(m_1)\Theta(ma)\overline{\tau}(m_2), \qquad m_1, m_2, m \in M,\ a \in A_\mathfrak{p}. \tag{9.8}$$

If ς is in $\mathfrak{Z}_1$

$$\Theta(ma; \varsigma) = \lim_{t\to\infty} e^{-t\Gamma(H_0)}\Phi^0(ma \exp tH_0; \varsigma).$$

By Lemma 15 this expression is equal to

$$\Gamma(\varsigma)\Theta(ma) + \lim_{t\to\infty} e^{-t\Gamma(H_0)}E^0\Psi_\varsigma(ma \cdot \exp tH_0).$$

Now, the last term in this formula approaches 0 as t approaches ∞ by Lemmas 16 and 21. Therefore

$$\Theta(ma;\varsigma) = \Gamma(\varsigma)\Theta(ma). \tag{9.9}$$

LEMMA 24. *For any fixed h in $\mathfrak{A}_{\mathfrak{p}}$ there are polynomials p and q such that for any $t \geqslant 0$,*

$$|e^{-t\Gamma(H_0)}\Phi^0(\exp tH_0; h) - \Theta(1;h)| \leqslant p(|\lambda|)q(|\tau|)e^{-2\varepsilon t}.$$

PROOF. Using the definition of Θ and the formula (9.7) we see that

$$\begin{aligned}\Theta(1;h) &= e^{-T\Gamma(H_0)}\Theta(\exp TH_0; h)\\ &= e^{-T\Gamma(H_0)}\Phi^0(\exp TH_0;h) + \int_T^\infty e^{-t\Gamma(H_0)}E^0\Psi(\exp tH_0;h)\,dt.\end{aligned}$$

The lemma then follows from Lemmas 16 and 21. □

COROLLARY. *For any fixed h in $\mathfrak{A}_{\mathfrak{p}}$ there are polynomials p and q such that for any $t \geqslant 0$*

$$|\Phi(\exp tH_0;h) - \Theta(\exp tH_0;h)| \leqslant p(|\lambda|)q(|\tau|)e^{-\varepsilon t}.$$

PROOF. We see that

$$\begin{aligned}&|\Phi(\exp tH_0;h) - \Theta(\exp tH_0;h)|\\ &\qquad\leqslant |\Phi^+(\exp tH_0;h)| + |\Phi^-(\exp tH_0;h)|\\ &\qquad\quad + |e^{t\Gamma(H_0)}E^0|\cdot|e^{-t\Gamma(H_0)}\Phi^0(\exp tH_0;h) - \Theta(1;h)|.\end{aligned}$$

The corollary then follows from Lemmas 21, 22, 23, and 24. □

It is clear from the definition of Θ that for any a in $A_{\mathfrak{p}}$

$$E^0\Theta(a) = \Theta(a). \tag{9.10}$$

For a in $A_{\mathfrak{p}}$, and $1 \leqslant j \leqslant r$, let $\theta_j(a)$ equal the vector $t_j(\Theta(a))$. Then

$$\Theta(a) = \sum_{1\leqslant j\leqslant r} \theta_j(a)\otimes\eta_j^{**}.$$

Write θ_1 as θ.

LEMMA 25. *For any nonnegative integer n there are polynomials p and q such that for any $t \geqslant 0$*

$$\left|\left(\frac{d}{dt}\right)^n [e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0) - \theta(\exp tH_0)]\right| \leqslant p(|\lambda|)q(|\tau|)e^{-\varepsilon t}.$$

PROOF. The expression

$$\left(\frac{d}{dt}\right)^n [e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0) - \theta(\exp tH_0)]$$

is equal to

$$t_1(\Phi(\exp tH_0;H_0^n) - \Theta(\exp tH_0;H_0^n)).$$

Now for any u in $\mathscr{V}_\tau$, $|t_1(u)| \leqslant |u|$. Our lemma then follows from the corollary to Lemma 24. □

LEMMA 26. *If ϕ_λ^τ is in $L^2(G)\otimes V_\tau$, θ is equal to zero.*

PROOF. The function $D(t)$ was defined in §3. Let

$$S^+ = \{t \geqslant 0\colon D(t) \leqslant \tfrac{1}{2}e^{2t\rho(H_0)}\}.$$

S^+ is compact. Let $\mathbf{R}^+ = \{t \geqslant 0\}$. Break up the integral

$$\int_0^\infty |e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0)|^2\,dt$$

into the sum of the integral over S^+ and the integral over $\mathbf{R}^+ - S^+$. The integral over S^+ is finite since ϕ_λ^τ is continuous. Now

$$\begin{aligned}\int_{\mathbf{R}^+-S^+} |e^{t\rho(H_0)}\phi_\lambda^\tau(\exp tH_0)|^2\,dt &\leqslant 2\cdot\int_{\mathbf{R}^+-S^+} |D(t)|\cdot|\phi_\lambda^\tau(\exp tH_0)|^2\,dt \\ &\leq 2\int_0^\infty D(t)\cdot|\phi_\lambda^\tau(\exp tH_0)|^2\,dt.\end{aligned}$$

By (3.5) this last integral equals $(2/c)\int_G |\phi_\lambda^\tau(x)|^2\,dx$, which is finite. Therefore,

$$\int_0^\infty e^{t\rho(H_0)}|\phi_\lambda^\tau(\exp tH_0)|^2\,dt < \infty.$$

Therefore, by Lemma 25

$$\int_0^\infty |\theta(\exp tH_0)|^2\,dt < \infty.$$

Now $\Theta(\exp tH_0) = e^{t\Gamma(H_0)}\Theta(1)$. But by (9.10), $\Theta(1)$ is nonzero only on the subspace of $\mathscr{V}_\tau$ spanned by eigenvectors of $\Gamma(H_0)$ that are associated with purely imaginary eigenvalues. Therefore, θ must be zero. □

10. Completion of the proof of Theorem 3′(a). We would like to apply the results of §9 to the collection $\{\phi_\omega^\tau(x)\colon \omega\in\mathscr{E}_d,\ \tau\in\mathscr{E}_K^2\}$ of τ-spherical functions which are derived from the matrix elements of square-integrable representations. That is,

$$\phi(x) = \phi_\omega^\tau(x) = (\dim\tau)^{-1/2}\sum_{ij}\xi_{1j}\otimes\xi_{2j}^*(\Phi_{1i},\pi_\omega(x)\Phi_{2j}) \tag{10.1}$$

in the notation of (5.1). Here π_ω is a square-integrable representation in the class of ω, acting on the Hilbert space $\mathscr{H}_\omega$. If $\tau=(\tau_1,\tau_2)$ then Φ_{1i} and Φ_{2j} are unit vectors in $\mathscr{H}_\omega$ that transform under $\pi_\omega|_K$ according to τ_1 and τ_2 respectively. $\{\xi_{1i}\}$ and $\{\xi_{2j}\}$ are orthonormal bases for the spaces on which τ_1 and τ_2 act, so $\{\xi_{1i}\otimes\xi_{2j}^*\}$ is an orthonormal base for V_τ, the space on which τ acts. Therefore

$$|\phi_\omega^\tau(x)|^2 = (\dim\tau)^{-1}\sum_{ij}|(\Phi_{1i},\pi_\omega(x)\Phi_{2j})|^2. \tag{10.2}$$

It follows that

$$|\phi_\omega^\tau(x)| \leqslant 1, \qquad x\in G. \tag{10.3}$$

From the Schur orthogonality relations on G, (2.7) and (2.8), we see that

$$\int_G |\phi^\tau_\omega(x)|^2\,dx = \beta(\omega)^{-1}. \tag{10.4}$$

Now we need to establish the estimate (9.1) for our collection $\{\phi^\tau_\omega\}$. Fix g_1, g_2 in $\mathscr{B}$. We use (10.1) and Lemma 13 to obtain an expression for $\phi^\tau_\omega(g_1;x;g_2)$. We obtain polynomials p and q, orthogonal sets of vectors $\{\Psi_{1\alpha}\}_{1\leqslant\alpha\leqslant t_1}$, $\{\Psi_{2\beta}\}_{1\leqslant\beta\leqslant t_2}$ in $\mathscr{H}_\omega$, and representations $\{\tau_{1\alpha}\}_{1\leqslant\alpha\leqslant t_1}$, $\{\tau_{2\beta}\}_{1\leqslant\beta\leqslant t_2}$ in $\mathscr{E}_K$ that satisfy the conditions of Lemma 13 and such that

$$|\phi^\tau_\omega(g_1;x;g_2)|^2 = \sum_{\alpha=1}^{t_1}\sum_{\beta=1}^{t_2} |(\Psi_{1\alpha}, \pi_\omega(x)\Psi_{2\beta})|^2. \tag{10.5}$$

Suppose $\omega = \omega(\lambda)$ for some λ in L'. Then the infinitesimal character of π_ω is χ_{λ_y}. (8.2) tells us that $|\lambda_y|^2 = |\omega|^2$. Therefore from (10.5) and the conditions of Lemma 13 we obtain the inequality

$$|\phi^\tau_\omega(g_1;x;g_2)|^2 \leqslant t_1t_2\cdot p(|\omega|)\cdot q(|\tau|) \leqslant p(|\omega|)\cdot q(|\tau|)^3 \tag{10.6}$$

for any x in G. From (2.8) and (10.5) we also see that

$$\int_G |\phi^\tau_\omega(g_1;x;g_2)|^2\,dx \leqslant \beta(\omega)^{-1}\cdot p(|\omega|)\cdot q(|\tau|)^3, \qquad x \bullet G. \tag{10.7}$$

LEMMA 27. *There are polynomials p and q such that for any x in G,*

$$|\phi^\tau_\omega(x)| < p(|\omega|)\cdot q(|\tau|)\Xi(x).$$

PROOF. Recall that r_1 and r_2 denote the number of roots in P_+ which when restricted to $\mathfrak{a}_\mathfrak{p}$ are respectively equal to μ_0 and $2\mu_0$. Then $\rho(H_0) = \frac{1}{2}(r_1+2r_2)$. Also

$$B(H_0,H_0) = 2(r_1+4r_2) = r^2.$$

Define $D(t)$ as in (3.5). Define a set S by

$$S = \{t\in\mathbf{R}\colon |D(t)| \leqslant \tfrac{1}{2}e^{2|t|\rho(H_0)}\}.$$

S is bounded. Define a positive infinitely differentiable function η on $\mathbf{R}$ such that for t not in S

$$\eta(t) = \tfrac{1}{2}e^{2|t|\rho(H_0)}.$$

Let $h(t)$ be the function

$$\phi^\tau_\omega(\exp tH_0)\cdot\eta^{1/2}(t).$$

Clearly $h(t)$ is infinitely differentiable. We wish to show that $h(t)$ is in $\mathscr{S}(\mathbf{R})\otimes V_\tau$, the tensor product of the Schwartz space of $\mathbf{R}$ with V_τ, the Hilbert space on which τ acts.

For any nonnegative integer n

$$\left(\frac{d}{dt}\right)^n h(t) = \sum_i \frac{n!}{(n-i)!(i)!}\left(\frac{d}{dt}\right)^i \phi^\tau_\omega(\exp tH_0)\cdot\left(\frac{d}{dt}\right)^{n-i}\eta^{1/2}(t).$$

There exist constants C^i such that

$$\left|\left(\frac{d}{dt}\right)^{n-i} \eta^{1/2}(t)\right| \leqslant C^i e^{|t|\rho(H_0)}$$

for all t not in S and for $0 \leqslant i \leqslant n$. Therefore since $\eta^{1/2}(t)$ is differentiable, there is a constant C_n such that

$$\left|\left(\frac{d}{dt}\right)^{n-i} \eta^{1/2}(t)\right| \leqslant C_n e^{|t|\rho(H_0)}$$

for all real t and for $0 \leqslant i \leqslant n$. Now ϕ_ω^τ is in $\mathscr{C}(G) \otimes V_\tau$ by [**3**(1), Lemma 65, Corollary 1], so for each s there is a constant C_ω^τ such that

$$|\phi_\omega^\tau(\exp tH_0; H_0^i)| \leqslant C_\omega^\tau \cdot \Xi(\exp tH_0) \cdot (1+\sigma(tH_0))^{-s}$$

for t in $\mathbf{R}$ and $0 \leqslant i \leqslant n$. But

$$1+\sigma(\exp tH_0) = 1 + r|t|, \qquad t \in \mathbf{R}.$$

By (4.1)

$$\Xi(\exp tH_0) \leqslant c(1+r|t|)^d e^{-|t|\rho(H_0)}, \qquad t \in \mathbf{R}.$$

Therefore for all real t,

$$\left|\left(\frac{d}{dt}\right)^{n} h(t)\right| \leqslant \left(\sum_i \frac{n!}{(n-i)!(i)!}\right) \cdot c \cdot C_\omega^\tau \cdot C_n (1+r|t|)^{-s+d}.$$

This proves that h is in $\mathscr{S}(\mathbf{R}) \otimes V_\tau$.

We shall take the Fourier transform of $h(t)$ on $\mathbf{R}$. Write

$$\hat{h}(s) = \int_{-\infty}^{\infty} h(t) e^{-ist}\, dt, \qquad s \in \mathbf{R}.$$

For any real t

$$\begin{aligned} |h(t)| &= (2\pi)^{-1} \left| \int_{-\infty}^{\infty} \hat{h}(s) e^{ist}\, ds \right| \\ &\leqslant (2\pi)^{-1} \left| \int_{-\infty}^{\infty} \hat{h}(s) e^{ist}(1+is)\cdot(1+is)^{-1}\, ds \right| \\ &\leqslant (2\pi)^{-1} \left(\int_{-\infty}^{\infty} |\hat{h}(s)(1+is)|^2\, ds \right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \frac{ds}{1+s^2} \right)^{1/2}. \end{aligned}$$

If we write h' for the derivative of h, this last expression equals $(\pi)^{-1/2}\|\hat{h}+\hat{h}'\|_2$, which in turn is equal to $(\pi)^{-1/2}\|h+h'\|_2$. Therefore for any real t

$$|h(t)| \leqslant \pi^{-1/2}(\|h\|_2 + \|h'\|_2).$$

We have used the fact that all the above integrals are absolutely convergent. This is true because both h and $\hat{h}$ are in $\mathscr{S}(\mathbf{R}) \otimes V_\tau$.

We see that it is necessary to estimate both $\|h\|_2$ and $\|h'\|_2$. Clearly $\|h\|_2$ is equal to

$$\int_S |\phi_\omega^\tau(\exp tH_0)|^2 \eta(t)\, dt + \int_{\mathbf{R}-S} |\phi_\omega^\tau(\exp tH_0)|^2 \eta(t)\, dt.$$

Now $\eta(t)$ is bounded on S, so by (10.3) there is a constant C_1 independent of ω and τ, such that

$$\int_S |\phi_\omega^\tau(\exp tH_0)|^2 \eta(t)\, dt \leqslant C_1.$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbf{R}-S} |\phi_\omega^\tau(\exp tH_0)|^2 \eta(t)\, dt \\
&\qquad \leqslant 2\int_{\mathbf{R}-S} |\phi_\omega^\tau(\exp tH_0)|^2 \cdot |D(t)|\, dt \\
&\qquad \leqslant 2\int_{\mathbf{R}} |\phi_\omega^\tau(\exp tH_0)|^2 \cdot |D(t)|\, dt \\
&\qquad \leqslant 2\int_{K\times K}\int_{\mathbf{R}} |\phi_\omega^\tau(k_1 \cdot \exp tH_0 \cdot k_2)|^2 \cdot |D(t)|\, dt\, dk_1\, dk_2
\end{aligned}$$

By (3.5) this last integral equals

$$\frac{4}{c}\int_G |\phi_\omega^\tau(x)|^2\, dx.$$

This expression equals $(\frac{4}{c})\beta(\omega)^{-1}$ by (10.4). Since $\beta(\omega)^{-1}$ is bounded independently of ω, by Lemma 3, $\|h\|_2$ is bounded independently of ω and τ.

Now we shall estimate $\|h'\|_2$. First of all we need a bound on the expression

$$\int_{\mathbf{R}} |\phi_\omega^\tau(\exp tH_0; H_0)|^2 \cdot |D(t)|\, dt.$$

For any x in G we have the following formula by (10.5)

$$|\phi_\omega^\tau(x; H_0)|^2 = \sum_{\alpha=1}^{t_1}\sum_{\beta=1}^{t_2} |(\Psi_{1\alpha}, \pi_\omega(x)\Psi_{2\beta})|^2$$

where $\{\Psi_{1\alpha}\}$, $\{\Psi_{2\beta}\}$, t_1 and t_2 satisfy the conditions of Lemma 13. Define

$$f_{\alpha,\beta}(x) = (\Psi_{1\alpha}, \pi_\omega(x)\Psi_{2\beta}).$$

By (10.7) and Lemma 3, there are polynomials p_1 and q_1 such that

$$\int_G \sum_{\alpha,\beta} |f_{\alpha,\beta}(x)|^2\, dx \leqslant p_1(|\omega|) \cdot q_1(|\tau|).$$

Let $\tau_{\alpha,\beta}$ be the double representation $(\tau_{1\alpha}, \tau_{2\beta})$ of K. As we did in §5, we can associate a $\tau_{\alpha,\beta}$-spherical function $\phi_{\alpha,\beta}$ to each $f_{\alpha,\beta}$. (Unlike the situation in §5, $\Psi_{1\alpha}$ and $\Psi_{2\beta}$ are not of norm 1.) Formula (10.2) and the Schur orthogonality relations, (2.7) and (2.8), can be used to derive

$$\int_G |f_{\alpha,\beta}(x)|^2\, dx = \int_G |\phi_{\alpha,\beta}(x)|^2\, dx.$$

Using this and formulas (10.2) and (3.5), we obtain the following inequality:

$$\begin{aligned}
&\int_{\mathbf{R}} |f_{\alpha,\beta}(\exp tH_0)|^2 \cdot |D(t)|\, dt \\
&\qquad \leqslant \dim \tau_{1\alpha} \cdot \dim \tau_{2\beta} \cdot \int_{\mathbf{R}} |\phi_{\alpha,\beta}(\exp tH_0)|^2 \cdot |D(t)|\, dt \\
&\qquad = \dim \tau_{1\alpha} \cdot \dim \tau_{2\beta} \cdot \int_{K\times K}\int_{\mathbf{R}} |\phi_{\alpha,\beta}(k_1 \cdot \exp tH_0 \cdot k_2)|^2 \cdot |D(t)|\, dt\, dk_1\, dk_2 \\
&\qquad = (2/c) \dim \tau_{1\alpha} \cdot \dim \tau_{2\beta} \cdot \int_G |\phi_{\alpha,\beta}(x)|^2\, dx \\
&\qquad = (2/c) \dim \tau_{1\alpha} \cdot \dim \tau_{2\beta} \cdot \int_G |f_{\alpha,\beta}(x)|^2\, dx.
\end{aligned}$$

Now the dimensions of $\tau_{1\alpha}$ and $\tau_{2\beta}$ are bounded by polynomials in $|\tau_{1\alpha}|$ and $|\tau_{2\beta}|$, which in turn are bounded by polynomials in $|\tau|$ by Lemma 13. Therefore there are polynomials p_2 and q_2 such that

$$\begin{aligned}
&\int_{\mathbf{R}} |\phi^\tau_\omega(\exp tH_0; H_0)|^2 \cdot |D(t)|\, dt \\
&\qquad \leqslant (2/c) p_2(|\omega|) \cdot p_2(|\tau|) \cdot \int_G \sum_{\alpha,\beta} |f_{\alpha,\beta}(x)|^2\, dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbf{R}} |\phi^\tau_\omega(\exp tH_0; H_0)|^2 |D(t)|\, dt \\
&\qquad \leqslant (2/c) p_1(|\omega|) \cdot p_2(|\omega|) \cdot q_1(|\tau|) \cdot q_2(|\tau|).
\end{aligned} \tag{10.8}$$

We can now estimate $\|h'\|_2$. This norm is the sum of the following two expressions:

(i)

$$\left(\int_{\mathbf{R}} \left| \frac{d}{dt} \phi^\tau_\omega(\exp tH_0) \right|^2 \cdot \eta(t)\, dt \right)^{1/2}.$$

(ii)

$$\left(\int_{\mathbf{R}} |\phi^\tau_\omega(\exp tH_0)|^2 \cdot \left| \frac{d}{dt} \eta^{1/2}(t) \right|^2 dt \right)^{1/2}.$$

There is a constant c_2 such that

$$\left| \frac{d}{dt} \eta^{1/2}(t) \right|^2 \leqslant c_2 \eta(t).$$

Therefore the second expression is bounded by $c_2\|h\|_2$. We break up the integral in the first expression into integrals over $\mathbf{R} - S$ and S. The integral over S causes no problem because $\eta(t)$ is bounded on S, and by (10.6), $|(d/dt)\phi^\tau_\omega(\exp tH_0)|$ is bounded by a polynomial in $|\omega|$ and $|\tau|$. On the other hand,

$$\begin{aligned}
\int_{\mathbf{R}-S} \left| \frac{d}{dt} \phi^\tau_\omega(\exp tH_0) \right|^2 \cdot \eta(t)\, dt &\leqslant 2 \int_{\mathbf{R}-S} \left| \frac{d}{dt} \phi^\tau_\omega(\exp tH_0) \right|^2 \cdot |D(t)|\, dt \\
&\leqslant 2 \int_{\mathbf{R}} \left| \frac{d}{dt} \phi^\tau_\omega(\exp tH_0) \right|^2 \cdot |D(t)|\, dt.
\end{aligned}$$

By (10.8) this last expression is bounded by a polynomial in $|\omega|$ and $|\tau|$. Therefore there are polynomials p_3 and q_3 such that

$$\|h'\|_2 \leqslant p_3(|\omega|) \cdot q_3(|\tau|).$$

We have now shown that there are polynomials p_4 and q_4 such that

$$|h(t)| \leqslant p_4(|\omega|) \cdot q_4(|\tau|), \qquad t \in \mathbf{R}.$$

This implies the inequality

$$|\phi_\omega^\tau(\exp tH_0)| \leqslant \eta^{-1/2}(t) p_4(|\omega|) \cdot q_4(|\tau|).$$

There is a constant c_3 such that

$$\eta^{-1/2}(t) \leqslant c_3 e^{-|t|\rho(H_0)}, \qquad t \in \mathbf{R}.$$

Also, by (4.1) and (4.3),

$$e^{-|t|\rho(H_0)} \leqslant \Xi(\exp tH_0), \qquad t \in \mathbf{R}.$$

Therefore, we can choose polynomials p and q such that

$$|\phi_\omega^\tau(\exp tH_0)| \leqslant p(|\omega|) \cdot q(|\tau|) \cdot \Xi(\exp tH_0), \qquad t \in \mathbf{R}.$$

However, each x in G is of the form $k_1 \cdot \exp tH_0 \cdot k_2$ for k_1 and k_2 in K and t in $\mathbf{R}$. Therefore

$$\begin{aligned} |\phi_\omega^\tau(x)| &= |\tau_1(k_1)\phi_\omega^\tau(\exp tH_0)\tau_2(k_2)| = |\phi_\omega^\tau(\exp tH_0)| \\ &\leqslant p(|\omega|) \cdot q(|\tau|) \cdot \Xi(\exp tH_0) \\ &= p(|\omega|) \cdot q(|\tau|) \cdot \Xi(x). \end{aligned}$$

This proves our lemma. □

COROLLARY. *For any g_1 and g_2 in $\mathscr{B}$ there are polynomials p and q such that*

$$|\phi_\omega^\tau(g_1; x; g_2)| \leqslant p(|\omega|) \cdot q(|\tau|) \cdot \Xi(x), \qquad x \in G.$$

PROOF. By (10.5) we have the formula

$$|\phi_\omega^\tau(g_1; x; g_2)|^2 = \sum_{\alpha=1}^{t_1} \sum_{\beta=1}^{t_2} |(\Psi_{1\alpha}, \pi_\omega(x)\Psi_{2\beta})|^2.$$

Let $\phi_{\alpha,\beta}$ be the $(\tau_{1\alpha}, \tau_{2\beta})$-spherical function associated with $(\Psi_{1\alpha}, \pi_\omega(x)\Psi_{2\beta})$. Apply Lemma 27 to $\phi_{\alpha,\beta}$. The proof of the corollary then follows from the conditions in Lemma 13. □

This corollary verifies the inequality (9.1) for our collection $\{\phi_\omega^\tau\}$.

Each λ in L_y' is regular, so Assumption 1 of §9 holds. To see that Assumption 2 is valid, we must look at the eigenvalues of the linear transformation $\Gamma(H_0)$ defined in §9. ($\Gamma(H_0)$, of course, depends on ω.) We see from Lemma 17, using the notation there, that the set of eigenvalues of $\Gamma(H_0)$ is equal to

$$\{\langle s_i\lambda, H_0\rangle : \lambda \in L_y',\ 1 \leqslant i \leqslant r\}.$$

Assumption 2 then is established by the following lemma.

LEMMA 28. *$\{\langle s_i\lambda, H_0\rangle : \lambda \in L'_y,\ 1 \leqslant i \leqslant r\}$ is a set of real numbers which generates a lattice in* $\mathbf{R}$.

PROOF. Any λ in L'_y assumes real values on the vector space $\mathfrak{a}_\mathfrak{p} + (-1)^{1/2}\mathfrak{a}_\mathfrak{k}$. Furthermore, any element in W maps the real vector space $\mathfrak{a}_\mathfrak{p} + (-1)^{1/2}\mathfrak{a}_\mathfrak{k}$ onto itself. $\langle s_i\lambda, H_0\rangle$ is equal to $\lambda(s_i^{-1}H_0)$, which is then a real number.

Let $\widetilde{L_y}$ be the lattice of functions from $\mathfrak{a}_\mathfrak{c}$ into $\mathbf{C}$ generated by the positive roots P. $\widetilde{L_y}$ is invariant under the action of W. If μ is in $\widetilde{L_y}$, the real number $\langle s_i\mu, H_0\rangle$ is an integer because for any α in P, $\alpha(H_0)$ is either 0, 1, or 2. But $\widetilde{L}_y$ is of finite index in L_y since G has finite center. This proves the lemma. □

We have shown that we can apply the results of §9 to our collection $\{\phi^\tau_\omega\}$. By Lemma 26, the function θ defined in §9 is zero. Then by Lemma 25 there are polynomials p and q such that

$$|\phi^\tau_\omega(\exp tH_0)| \leqslant p(|\omega|)\cdot q(|\tau|)\cdot e^{-t\rho(H_0)}e^{-\varepsilon t}, \qquad t \geqslant 0.$$

Therefore, for any $s \geqslant 0$ there are polynomials p and q such that

$$|\phi^\tau_\omega(\exp tH_0)| \leqslant p(|\omega|)\cdot q(|\tau|)\cdot e^{-t\rho(H_0)}(1+rt)^{-s}, \qquad t \geqslant 0.$$

But for $t \geqslant 0$ we have the formulae

$$e^{-t\rho(H_0)} \leqslant \Xi(\exp tH_0), \qquad 1+\sigma(\exp tH_0) = 1+rt.$$

Now each x in G is of the form $k_1\cdot\exp tH_0\cdot k_2$ for $t \geqslant 0$ and k_1 and k_2 in K. But ϕ^τ_ω is τ-spherical and Ξ and σ are bi-invariant under K. We have proved the following lemma

LEMMA 29. *For any x in G*

$$|\phi^\tau_\omega(x)| \leqslant p(|\omega|)\cdot q(|\tau|)\cdot \Xi(x)(1+\sigma(x))^{-s}. \ \square$$

COROLLARY. *For any g_1 and g_2 in $\mathscr{B}$ there are polynomials p and q such that for any x in G*

$$|\phi^\tau_\omega(g_1; x; g_2)| \leqslant p(|\omega|)\cdot q(|\tau|)\cdot \Xi(x)(1+\sigma(x))^{-s}.$$

PROOF. The proof follows from Lemma 29 in the same way as the corollary to Lemma 27 followed from Lemma 27. □

We have proved Theorem 3′(a).

11. Application to the continuous series. Let τ be an irreducible unitary double representation of K on the Hilbert space V_τ. Recall from §5 that

$$L^\tau = \bigoplus_{\sigma\in\mathscr{E}_M} L^\tau_\sigma$$

is an orthogonal direct sum. For any fixed τ, $L^\tau_\sigma = 0$ for all but a finite number of σ.

We would like to apply the results of §9 to the collection $\{E_\Lambda(\psi^\tau_\sigma : x)\}$ of Eisenstein integrals. Λ is to range over the nonzero reals, and τ and σ will range over $\mathscr{E}^2_K$ and $\mathscr{E}_M$ respectively. ψ^τ_σ will be any unit vector in L^τ_σ.

We will have to check the estimate (9.1) and Assumptions 1 and 2 of §9 for our collection $\{E_\Lambda(\psi^\tau_\sigma : x)\}$. Then we will examine the functions $\theta = \theta_\Lambda(\psi^\tau_\sigma : ma)$ associated with $E_\Lambda(\psi^\tau_\sigma : x)$ in §9.

LEMMA 30. *Fix g_1, g_2 in $\mathscr{B}$. Then for fixed x in G, $E_\Lambda(\psi_\sigma^\tau\colon g_1; x; g_2)$ can be regarded as an entire function of Λ. Also, there are polynomials p_1, p_2, and q, dependent only on g_1 and g_2, such that for every integer $n \geqslant 0$, for $t \geqslant 0$, and for Λ in $\mathbf{C}$,*

$$\left|\left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi_\sigma^\tau\colon g_1; \exp tH_0; g_2)\right| \leqslant e^{|\Lambda_I|t} \cdot p_1(|\sigma|) \cdot p_2(|\Lambda|)| \cdot q(|\tau|) \cdot \Xi(\exp tH_0) \cdot t^n.$$

(*We write Λ_I for the imaginary part of the complex number Λ.*)

PROOF. We have the formula (5.5):

$$E_\Lambda(\psi_\sigma^\tau\colon x) = \int_K \psi_\sigma^\tau(xk)\tau(k^{-1})e^{(i\Lambda\mu_0-\rho)(H(xk))}\,dk.$$

For fixed x, this is clearly an entire function of Λ. Derivatives of $E_\Lambda(\psi_\sigma^\tau\colon x)$ by means of left or right invariant differential operators are entire functions in Λ.

From (5.6) we obtain the formula

$$\left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi_\sigma^\tau\colon x) = (t_1t_2)^{-1/2}\sum_{ij}\xi_{1i}\otimes\xi_{2j}^* \cdot \left(\frac{d}{d\Lambda}\right)^n (\Phi_{1i}, \pi_{\sigma,\Lambda}(x)\Phi_{2j}),$$

where t_1t_2 is the dimension of the representation τ. We apply Lemma 13 to each of the functions $(\Phi_{1i}, \pi_{\sigma,\Lambda}(x)\Phi_{2j})$. As a result we obtain polynomials p, q, orthogonal sets of vectors

$$\{\Psi_{1\alpha}\colon 1 \leqslant \alpha \leqslant t_1\}, \qquad \{\Psi_{2\beta}\colon 1 \leqslant \beta \leqslant t_2\}$$

in $\mathscr{H}_{\sigma,\Lambda}$, and the representations

$$\{\tau_{1\alpha}\colon 1 \leqslant \alpha \leqslant t_1\}, \qquad \{\tau_{2\beta}\colon 1 \leqslant \beta \leqslant t_2\}$$

in $\mathscr{E}_K$ that satisfy the conditions of Lemma 13. In addition

$$\left|\left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi_\sigma^\tau\colon g_1; x; g_2)\right|^2 = \sum_{\alpha=1}^{t_1}\sum_{\beta=1}^{t_2}\left|\left(\frac{d}{d\Lambda}\right)^n (\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(x)\Psi_{2\beta})\right|^2. \tag{11.1}$$

For any (α, β), the vectors $\Psi_{1\alpha}$ and $\Psi_{2\beta}$ transform under $\pi_{\sigma,\Lambda}|_K$ according to the representations $\tau_{1\alpha}$ and $\tau_{2\beta}$ respectively. Let $\tau_{\alpha,\beta}$ be the double representation $(\tau_{1\alpha}, \tau_{2\beta})$ of K. Fix a vector $\psi_{\alpha,\beta}$ in $L_\sigma^{\tau_{\alpha,\beta}}$ such that $E_\Lambda(\psi_{\alpha\beta}\colon x)$ is the $\tau_{\alpha,\beta}$-spherical function associated with the function $(\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(x)\Psi_{2\beta})$. By (5.6) we have the inequality

$$\left|\left(\frac{d}{d\Lambda}\right)^n (\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(\exp tH_0)\Psi_{2\beta})\right| \leqslant (\dim\tau_{\alpha,\beta})^{1/2}\cdot\left|\left(\frac{d}{d\Lambda}\right)^n E_\Lambda(\psi_{\alpha,\beta}\colon \exp tH_0)\right|.$$

By (5.5), the right-hand side of this inequality equals

$$(\dim\tau_{\alpha,\beta})^{1/2}.\left|\left(\frac{d}{d\Lambda}\right)^n\int_K \psi_{\alpha,\beta}(\exp tH_0\cdot k)\tau_{\alpha,\beta}(k^{-1})\cdot e^{(i\Lambda\mu_0-\rho)(H(\exp tH_0\cdot k))}\,dk\right|,$$

and this expression equals

$$(\dim \tau_{\alpha,\beta})^{1/2} \cdot \left| \int_K \psi_{\alpha,\beta}(\exp tH_0 \cdot k)\tau_{\alpha,\beta}(k^{-1}) \cdot \mu_0(H(\exp tH_0 \cdot k))^n \cdot e^{(i\Lambda\mu_0 - \rho)(H(\exp tH_0 \cdot k))}\, dk \right|.$$

Now there is a real t_k such that

$$H(\exp tH_0 \cdot k) = t_k H_0.$$

Then [**3(g)**, Lemma 35] and [**3(g)**, Lemma 35, Corollary 2] tell us that

$$|t_k| \leqslant t, \qquad t \geqslant 0,\ k \in K.$$

Therefore, if $t \geqslant 0$, we have the two inequalities

$$|\mu_0(H(\exp tH_0 \cdot k))|^n = |t_k|^n \leqslant t^n,$$

and

$$|e^{(i\Lambda\mu_0 - \rho)(H(\exp tH_0 \cdot k))}| \leqslant e^{|\Lambda_I|t}.$$

Notice also that

$$|\psi_{\alpha,\beta}(\exp tH_0 \cdot k)\tau_{\alpha,\beta}(k^{-1})| = |\psi_{\alpha,\beta}(1)|.$$

Therefore, for $t \geqslant 0$, we have the inequality

$$\begin{aligned} &\left| \left(\frac{d}{d\Lambda}\right)^n (\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(\exp tH_0)\Psi_{2\beta}) \right| \\ &\quad \leqslant (\dim \tau_{\alpha,\beta})^{1/2} \cdot |\psi_{\alpha,\beta}(1)| \cdot e^{|\Lambda_I|t} \cdot t^n \cdot \int_K e^{-\rho(H(\exp tH_0 \cdot k))}\, dk \\ &\quad = (\dim \tau_{\alpha,\beta})^{1/2} \cdot |\psi_{\alpha,\beta}(1)| \cdot e^{|\Lambda_I|t} \cdot t^n \cdot \Xi(\exp tH_0). \end{aligned}$$

But $|\psi_{\alpha,\beta}(1)| = \|\psi_{\alpha,\beta}\|_M$. From the remarks in §5 we see that

$$\|\psi_{\alpha,\beta}\|_M = (\dim \sigma)^{1/2} \cdot \|\Psi_{1\alpha}\| \cdot \|\Psi_{2\beta}\|.$$

However, by Lemma 13(i) and formula (8.3),

$$\|\Psi_{1\alpha}\|^2 \cdot \|\Psi_{2\beta}\|^2 \leqslant p((|\sigma|^2 + r^{-2}|\Lambda|^2)^{1/2})^2 \cdot q(|\tau|)^2.$$

Now $\dim \tau_{\alpha,\beta}$ and $\dim \sigma$ are bounded by polynomials in $|\tau_{\alpha,\beta}|$ and $|\sigma|$ respectively, by the Weyl dimension formula. But by Lemma 13, $|\tau_{\alpha,\beta}|$, t_1, and t_2 are all bounded by polynomials in $|\tau|$. The inequality in our lemma then follows from formula (11.1). □

Recall the definitions of $\tilde{\omega}^{\mathfrak{m}}$, L_1, and L_1' from §2. Let $\widetilde{L_1}$ be the lattice of real linear functionals on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ which is generated by the restrictions of roots of $(\mathfrak{g}, \mathfrak{a})$. Then $\widetilde{L_1} \subset L_1$ and $\widetilde{L_1}$ is a lattice of finite index in L_1. Recall that

$$q = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{k} - \operatorname{rank} \mathfrak{g} + \operatorname{rank} \mathfrak{k}) = \begin{cases} \tfrac{1}{2}[P_+] + 1 & \text{in Case I,} \\ \tfrac{1}{2}[P_+] & \text{in Case II.} \end{cases}$$

LEMMA 31. *There is a $\delta_1 > 0$ such that for any μ in L_1' the function*

$$\Lambda^{2q} \cdot \tilde{\omega}(-\mu - i\Lambda\mu_0)^{-1}$$

is holomorphic in the region $|\Lambda_I| < \delta_1$. *In addition, there exists a polynomial* p, *independent of* μ, *such that for* μ *in* L_1' *and* $|\Lambda_I| < \delta_1$,

$$|\Lambda^{2q} \cdot \tilde{\omega}(-\mu - i\Lambda\mu_0)^{-1}| \leqslant p(|\Lambda|).$$

PROOF. We have the formula

$$\tilde{\omega}(-\mu - i\Lambda\mu_0)^{-1} = \tilde{\omega}^{\mathrm{m}}(-\mu)^{-1} \cdot \prod_{\alpha \in P_+} \langle -\mu - i\Lambda\mu_0, H_\alpha\rangle^{-1}.$$

Note that $\Lambda^{2q} \cdot \tilde{\omega}(-\mu - i\Lambda\mu_0)^{-1}$ is a meromorphic function of Λ.

Since μ is in L_1', $\tilde{\omega}^{\mathrm{m}}(\mu)$ is not equal to zero. The numbers $\{|\tilde{\omega}^{\mathrm{m}}(-\mu)| \colon \mu \in L_1'\}$ are actually bounded away from zero. This can be seen by an argument similar to that used in the proof of Lemma 3.

If α is in P_+, the number $\langle -\mu - i\Lambda\mu_0, H_\alpha\rangle$ equals zero only if

$$\Lambda = i\mu(H_\alpha) \cdot \mu_0(H_\alpha)^{-1}.$$

It is well known that the numbers $\{\tilde{\mu}(H_\alpha) \colon \tilde{\mu} \in \widetilde{L_1}\}$ generate a lattice in $\mathbf{R}$. Therefore, since $\widetilde{L_1}$ is of finite index in L_1, the numbers

$$\{\mu(H_\alpha) \cdot \mu_0(H_\alpha)^{-1} \colon \mu \in L_1\}$$

generate a lattice in $\mathbf{R}$. Let ε_α be the positive generator of this lattice. Put

$$\delta_1 = \inf_{\alpha \in P_+} (\tfrac{1}{2}\varepsilon_\alpha).$$

Then for any α in P_+ and any μ in L_1', either

$$|\mu(H_\alpha) \cdot \mu_0(H_\alpha)^{-1}| = 0$$

or

$$|\mu(H_\alpha) \cdot \mu_0(H_\alpha)^{-1}| \geqslant 2\delta_1.$$

In either case, the function

$$\Lambda \cdot \langle -\mu - i\Lambda\mu_0, H_\alpha\rangle^{-1}$$

is holomorphic in the region $|\Lambda_I| < \delta_1$, and it is bounded independently of μ by a polynomial in $|\Lambda|$. Our lemma follows from the fact that $2q \geqslant [P_+]$. □

COROLLARY. *If* μ *is in* L_1' *and* Λ *is not equal to zero, then* $\tilde{\omega}(-\mu - i\Lambda\mu_0)$ *is not equal to zero.* □

LEMMA 32. *The real parts of the set*

$$\{\langle -\mu - i\Lambda\mu_0, sH_0\rangle \colon \mu \in L_1, \Lambda \in \mathbf{R}, s \in W\}$$

of complex numbers form a lattice in $\mathbf{R}$.

PROOF. The real part of $\langle -\mu - i\Lambda\mu_0, sH_0\rangle$ is equal to $-\langle s^{-1}\mu, H_0\rangle$. If μ is in $\widetilde{L_1}$, we can regard μ as an integral sum of roots of $(\mathfrak{g}, \mathfrak{a})$. Then $s^{-1}\mu$ is also an integral sum of roots of $(\mathfrak{g}, \mathfrak{a})$. If α is a root of $(\mathfrak{g}, \mathfrak{a})$, $\alpha(H_0)$ is an integer. Therefore $-\langle s^{-1}\mu, H_0\rangle$ is an integer for any μ in $\widetilde{L_1}$. Our lemma follows from the fact that $\widetilde{L_1}$ is of finite index in L_1. □

Lemma 30, the corollary to Lemma 31, and Lemma 32 verify the estimate (9.1) and Assumptions 1 and 2 of §9 for our collection $\{E_\Lambda(\psi_\sigma^\tau : x)\}$. We define the function $\theta_\Lambda = \theta_\Lambda(\psi_\sigma^\tau : ma)$ from $MA_{\mathfrak{p}}$ to V_τ as in §9. By Lemma 25, there exists, for every nonnegative integer n, polynomials p_1, p_2, q, and a number $\varepsilon > 0$, all independent of Λ, σ, and τ, such that for any $t \geqslant 0$,

$$\left|\left(\frac{d}{dt}\right)^n [e^{t\rho(H_0)}E_\Lambda(\psi_\sigma^\tau : \exp tH_0) - \theta_\Lambda(\psi_\sigma^\tau : \exp tH_0)]\right| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|)\cdot e^{-\varepsilon t}. \tag{11.2}$$

12. The linear transformations $c^+(\Lambda)$ and $c^-(\Lambda)$. Let us fix τ in $\mathscr{E}_K^2$ and σ in $\mathscr{E}_M$. Let μ be a real linear functional on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$ associated with σ. Fix ψ in L_σ^τ. We shall let $\Lambda \neq 0$ vary in $\mathbf{R}$. Put $\phi_\Lambda(x) = E_\Lambda(\psi : x)$. For m in M, and a in $A_{\mathfrak{p}}$, define $\Phi_\Lambda(ma) = \Phi_\Lambda(\psi : ma)$, $\Psi_\Lambda(ma) = \Psi_\Lambda(\psi : ma)$, and $\Theta_\Lambda(ma) = \Theta_\Lambda(\psi : ma)$ corresponding to the function ϕ_Λ as in §9.

The projections $\{E_{s_i}\}$ were defined in §9. Then

$$\Theta_\Lambda(ma) = E_{s_1}\Theta_\Lambda(ma) + \cdots + E_{s_r}\Theta_\Lambda(ma).$$

Let λ be the linear functional $-\mu - i\Lambda\mu_0$. If ς is in $\mathfrak{Z}_1$, then by Lemma 17

$$\Gamma(\varsigma)(E_{s_i}\Theta_\Lambda)(ma) = \langle\gamma_1(\varsigma), s_i\lambda\rangle(E_{s_i}\Theta_\Lambda)(ma). \tag{12.1}$$

Recall that $\bar{\tau}$ was the restriction of τ to M. By (9.8), Θ_Λ is $\bar{\tau}$-spherical. Let $\Theta_{\Lambda,i} = E_{s_i}\Theta_\Lambda$. Since the actions of $\bar{\tau}(M)$ and $\Gamma(\mathfrak{Z}_1)$ on the vector space $\mathscr{V}_\tau$ commute, and since the $\Theta_{\Lambda,i}$ are eigenvectors of $\Gamma(\varsigma)$, each $\Theta_{\Lambda,i}$ is a $\bar{\tau}$-spherical function.

Now suppose for some s_i that the linear functional $s_i\mu_0$ does not vanish on $\mathfrak{a}_{\mathfrak{k}}$. We will show that $\Theta_{\Lambda,i} = 0$. If ς_M is in $\mathfrak{Z}_M$, and m is in M, by (9.9) and (12.1) we see that

$$\Theta_{\Lambda,i}(m;\varsigma_M) = \Gamma(\varsigma_M)\Theta_{\Lambda,i}(m) = \langle\gamma_1(\varsigma_M), s_i\lambda\rangle\Theta_{\Lambda,i}(m). \tag{12.2}$$

Thus if $\Theta_{\Lambda,i}$ is regarded as a function on the compact group M, it is an eigenfunction of $\mathfrak{Z}_M$. The infinitesimal character corresponds to the restriction of the linear functional $s_i\lambda = -s_i(\mu + i\Lambda\mu_0)$ to $\mathfrak{a}_{\mathfrak{k}}$. Now $i\Lambda(s_i\mu_0)$ is real-valued on $\mathfrak{a}_{\mathfrak{k}}$, so that $s_i\lambda$ is not purely imaginary on $\mathfrak{a}_{\mathfrak{k}}$. However, it is well known that the eigenfunctions of a compact Lie group have eigenvalues corresponding to purely imaginary linear functionals on a Cartan subalgebra. Therefore $\Theta_{\Lambda,i}$ must be zero.

Then we can write

$$\Theta_\Lambda = \Theta_\Lambda^+ + \Theta_\Lambda^-.$$

Θ_Λ^+ and Θ_Λ^- are the sums of those $\Theta_{\Lambda,i}$ for which the Weyl group element s_i, when restricted to $\mathfrak{a}_{\mathfrak{p}}$, is respectively the identity or reflection about 0. Now by (9.7),

$$\Theta_{\Lambda,i}(m\exp tH_0) = e^{t\Gamma(H_0)}\Theta_{\Lambda,i}(m).$$

Therefore

$$\Theta_\Lambda^+(m\exp tH_0) = e^{i\Lambda t}\Theta_\Lambda^+(m),$$
$$\Theta_\Lambda^-(m\exp tH_0) = e^{-i\Lambda t}\Theta_\Lambda^-(m).$$

In the notation of §9, $\theta_\Lambda(ma) = t_1(\Theta_\Lambda(ma))$. Let $\theta_\Lambda^+(ma) = t_1(\Theta_\Lambda^+(ma))$, $\theta_\Lambda^-(ma) = t_1(\Theta_\Lambda^-(ma))$. Then

$$\theta_\Lambda(m \exp tH_0) = e^{i\Lambda t}\theta_\Lambda^+(m) + e^{-i\Lambda t}\theta_\Lambda^-(m).$$

Since the functions $\Theta_\Lambda^\pm(m)$ are $\bar{\tau}$-spherical, so are the functions $\theta_\Lambda^\pm(m)$. This means that the functions $\theta^\pm(m)$ are in L^τ. θ_Λ^+ depend linearly on the ψ in L_σ^τ that we fixed at the beginning of the section. We write this as $\theta_\Lambda^\pm = c^\pm(\Lambda)\psi$. $c^+(\Lambda)$ and $c^-(\Lambda)$ are linear transformations from L_σ^τ into L^τ. Since L^τ is an orthogonal direct sum of spaces L_σ^τ, we can extend the definition of $c^+(\Lambda)$ and $c^-(\Lambda)$ to all of L^τ. Then $c^\pm(\Lambda)$ are linear transformations of L^τ into itself.

For m in M and t in $\mathbf{R}$ we have the formula

$$\theta_\Lambda(\psi\colon m \exp tH_0) = e^{i\Lambda t}(c^+(\Lambda)\psi)(m) + e^{-i\Lambda t}(c^-(\Lambda)\psi)(m). \tag{12.3}$$

Suppose that the restriction of some s_i to $\mathfrak{a}_\mathfrak{p}$ is the identity. Then $\mathfrak{a}_\mathfrak{k}$ is an invariant subspace of s_i. It is known that s_i is in W_1. Since $\{s_1 = 1, s_2, \ldots, s_r\}$ is a set of representatives of cosets of W_1 in W, $s_i = s_1 = 1$. On the other hand, if the restriction of an element s_j to $\mathfrak{a}_\mathfrak{p}$ is a reflection, we can represent s_j as the nontrivial element in the group M'/M defined in §2. Therefore there is only one such s_j, which we shall denote by s_2. Therefore, if ψ is any vector in L_σ^τ,

$$(c^+(\Lambda)\psi)(m) = t_1(\Theta_{\Lambda,1}(m)),$$
$$(c^-(\Lambda)\psi)(m) = t_1(\Theta_{\Lambda,2}(m)).$$

Suppose that ψ is an eigenfunction of $\mathfrak{Z}_M$ in L^τ with infinitesimal character χ_μ^M, for some μ in L_1'. Then from (12.2), we see that the infinitesimal characters of $c^+(\Lambda)\psi$ and $c^-(\Lambda)\psi$ are χ_μ^M and $\chi_{\mu'}^M$, respectively. Here $\mu' = s_2\mu$.

For any μ in L_1' let $\mathscr{E}_M(\mu)$ be the set of all σ in $\mathscr{E}_M$ such that the linear functional associated to either σ or σ' is μ. It is clear that $|\sigma| = |\sigma'|$ for any σ in $\mathscr{E}_M$. Therefore, for any σ_1, σ_2 in $\mathscr{E}_M(\mu)$,

$$|\sigma_1| = |\sigma_2|. \tag{12.4}$$

Let L_μ^τ be the direct sum of all L_σ^τ for which σ is in $\mathscr{E}_M(\mu)$. From the above discussion we see that L_μ^τ is an invariant subspace of the linear transformations $c^\pm(\Lambda)$.

We wish to prove some estimates for $c^+(\Lambda)$ and $c^-(\Lambda)$. We need to examine the functions

$$t_i(\Psi_\Lambda)\colon MA_\mathfrak{p} \to V_\tau.$$

From the definition in §9,

$$t_i(\Psi_\Lambda(\psi\colon ma)) = d(ma)E_\Lambda(\psi\colon ma; u_i(H_0)'). \tag{12.5}$$

We shall allow Λ to assume complex values.

LEMMA 33. *Suppose ψ is in L_σ^τ, and $\|\psi\|_M = 1$. Then for fixed $t \geqslant 0$, $t_i(\Psi_\Lambda(\psi\colon \exp tH_0))$ is an entire function in Λ. Furthermore, there exist polynomials p_1, p_2, and q such that for Λ in $\mathbf{C}$ and $t \geqslant 0$,*

$$|t_i(\Psi_\Lambda(\psi\colon \exp tH_0))| \leqslant e^{-t(1-|\Lambda_I|)} \cdot (1+rt)^d \cdot p_1(|\sigma|) \cdot p_2(|\Lambda|) \cdot q(|\tau|).$$

Here $r^2 = B(H_0, H_0)$.

PROOF. It is clear from (12.5) and Lemma 30 that $t_i(\Psi_\Lambda(\psi\colon \exp tH_0))$ is an entire function of Λ.

Now, using the notation of the proof of Lemma 16, we see $|t_i(\Psi_\Lambda(\exp tH_0))|$ equals

$$\left|\sum_j e^{t\rho(H_0)} E_\Lambda(\psi\colon \exp tH_0; \theta(N_{ij})g_{ij}\eta_j')\right|.$$

If $t \geqslant 0$, this expression is bounded by

$$e^{-t}\left|e^{\rho(tH_0)}\sum_j E_\Lambda(\psi\colon \theta(N_{ij}); \exp tH_0; g_{ij}\eta_j')\right|.$$

Now by Lemma 30 there are polynomials p_1, p_2, and q such that for any $t \geqslant 0$

$$\begin{aligned}\left|\sum_j E_\Lambda(\psi\colon \theta(N_{ij}); \exp tH_0; g_{ij}\eta_j')\right|& \\ &\leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|)\cdot e^{|\Lambda_I|t}\cdot \Xi(\exp tH_0)\\ &\leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|)\cdot e^{|\Lambda_I|t}\cdot e^{-t\rho(H_0)}\cdot (1+rt)^d\end{aligned}$$

(this last inequality follows from (4.1)). Therefore $|t_i(\Psi_\Lambda(\exp tH_0))|$ is bounded by

$$e^{-(1-|\Lambda_I|t)}\cdot(1+rt)^d\cdot p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|). \quad \square$$

Let q be the integer in Lemma 31.

LEMMA 34. *Let δ be the minimum of δ_1 and $\frac{1}{4}$, where δ_1 is the positive constant in Lemma 31. Choose ψ in L_σ^τ with $\|\psi\|_M = 1$. Then if $1 \leqslant i, j \leqslant r$, the function $\Lambda^{2q}\cdot t_i(\Theta_{\Lambda,j}(\psi\colon 1))$ is analytic in the region $|\Lambda_I| < \delta$. In addition, there exist polynomials p_1, p_2, and q such that in the region $|\Lambda_I| < \delta$,*

$$|\Lambda^{2q}\cdot t_i(\Theta_{\Lambda,j}(\psi\colon 1))| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|).$$

PROOF. We have the formula

$$\Theta_\Lambda(1) = E^0\Phi_\Lambda(1) + \int_0^\infty e^{-t\Gamma(H_0)}E^0\Psi_\Lambda(\exp tH_0)\,dt.$$

We assume that $\Theta_{\Lambda,j}$ is not zero, so by our earlier remarks, $j = 1$ or 2. Then $s_jH_0 = \varepsilon_jH_0$ where $\varepsilon_j = 1$ or -1. E_{s_j} is the projection of $\mathscr{V}_\tau$ onto the eigenspace of $\Gamma(H_0)$ corresponding to the eigenvalue $\langle s_j(\mu + i\Lambda\mu_0), H_0\rangle$ where μ is a linear functional in L_1' associated with σ. This eigenvalue equals $i\varepsilon_j\Lambda$, which is purely imaginary. Therefore $E_{s_j}E^0 = E_{s_j}$, so that

$$\Lambda^{2q}\Theta_{\Lambda,j}(1) = \Lambda^{2q}E_{s_j}\Phi_\Lambda(1) + \Lambda^{2q}\int_0^\infty e^{-i\varepsilon_j\Lambda t}E_{s_j}\Psi_\Lambda(\exp tH_0)\,dt.$$

Now if $1 \leqslant l \leqslant r$

$$t_l(\Phi_\Lambda(1)) = E_\Lambda(\chi\colon 1; \eta_l').$$

By Lemma 30 there are polynomials p_1, p_2, q such that

$$|E_\Lambda(\psi\colon 1;\eta_l')| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|). \tag{12.6}$$

By the last lemma we can also choose polynomials p_1, p_2, q such that for any $t \geqslant 0$,

$$\begin{aligned} &|e^{-i\varepsilon_j\Lambda t}t_l(\Psi_\Lambda(\exp tH_0))| \\ &\qquad \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|)\cdot e^{-t(1-2|\Lambda_I|)}\cdot(1+rt)^d. \end{aligned} \tag{12.7}$$

Therefore, the integral

$$\int_0^\infty e^{-i\varepsilon_j\Lambda t}t_l(\Psi_\Lambda(\exp tH_0))\,dt$$

converges uniformly for Λ in compact subsets of $|\Lambda_I| < \frac{1}{4}$ and so is an analytic function of Λ in this region. Furthermore, by Lemmas 18 and 31, the matrices of the linear transformations $\Lambda^{2q}E_{s_j}$ relative to the basis $\{\eta_1^{**},\ldots,\eta_r^{**}\}$ have components which are analytic in $|\Lambda_I| < \delta_1$ and bounded by a polynomial in Λ in this region. Therefore, the functions $\Lambda^{2q}t_i(\Theta_{\Lambda,j}(1))$ are analytic in the region $|\Lambda_I| < \delta$. We see from (12.6) and (12.7) that it is possible to choose polynomials p_1, p_2, and q such that for $|\Lambda_I| < \delta$

$$|\Lambda^{2q}t_i(\Theta_{\Lambda,j}(1))| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|). \quad \square$$

COROLLARY. *Choose ψ in L_σ^τ with $\|\psi\|_M = 1$. Then the functions $\Lambda^{2q}c^\pm(\Lambda)\psi(1)$ are analytic in the region $|\Lambda_I| < \delta$. Furthermore, there are polynomials p_1, p_2, and q, independent of σ, Λ, and τ such that for $|\Lambda_I| < \delta$*

$$|\Lambda^{2q}c^\pm(\Lambda)\psi(1)| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|).$$

PROOF. The corollary follows from the lemma if we recall that the functions $c^+(\Lambda)\psi(1)$ and $c^-(\Lambda)\psi(1)$ equal $t_1(\Theta_{\Lambda,1}(1))$ and $t_1(\Theta_{\Lambda,2}(1))$ respectively. $\square$

Therefore, $c^\pm(\Lambda)\psi(1)$ is meromorphic in $|\Lambda_I| < \delta$, with the only possible pole being at $\Lambda = 0$. Assume it has a pole of order $N^\pm(\psi)$ at $\Lambda = 0$. Let us agree to write $N^\pm(\psi) = 0$ if $c^\pm(\Lambda)\psi$ has no poles at $\Lambda = 0$. Then

$$0 \leqslant N^\pm(\psi) \leqslant 2q.$$

The function

$$g^\pm(\psi\colon \Lambda) = \Lambda^{N^\pm(\psi)}c^\pm(\Lambda)\psi(1)$$

is holomorphic in the region $|\Lambda_I| < \delta$. For any Λ in this region we can write

$$g^\pm(\psi\colon \varsigma) = a_0^\pm + a_1^\pm(\varsigma-\Lambda) + a_2^\pm(\varsigma-\Lambda)^2 + \cdots \tag{12.8}$$

where ς remains in some neighborhood of Λ, and $a_0^\pm, a_1^\pm, a_2^\pm, \ldots$ are vectors in V_τ.

LEMMA 35. *For every nonnegative integer n there are polynomials p_1, p_2, q such that in the region $|\Lambda_I| < \delta/2$,*

$$\left|\left(\frac{d}{d\Lambda}\right)^n g^\pm(\psi\colon \Lambda)\right| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|).$$

PROOF. From (12.8), it is easy to show that

$$\left|\left(\frac{d}{d\Lambda}\right)^n g^{\pm}(\psi:\Lambda)\right| = \left|n!(2\pi i)^{-1}\cdot\int_\Gamma g^{\pm}(\psi:\xi)(\xi-\Lambda)^{-(n+1)}\,d\xi\right|$$
$$= n!(2\pi)^{-1}\left|\int_\Gamma \xi^{2q}c^{\pm}(\xi)\psi(1)\cdot(\xi-\Lambda)^{-(n+1)}\cdot\xi^{-2q+N^{\pm}(\chi)}\,d\xi\right|$$

where Γ is any curve in $|\xi_I|<\delta$ that winds around Λ once. Choose Γ such that for any ξ on Γ

(i) $\delta/2\leqslant|\xi-\Lambda|\leqslant 1$,

(ii) $\delta/2\leqslant|\xi|$.

This is clearly possible since $\delta<\frac{1}{4}$. Lemma 35 then follows from the corollary to Lemma 34. ◻

13. Relation between $c^{\pm}(\Lambda)$ and $\beta(\sigma,\Lambda)$. Let $\tau=(\tau_1,\tau_2)$ be an irreducible unitary double representation of K on the finite-dimensional Hilbert space V_τ. Let $\mathscr{E}$ be the finite-dimensional vector space of endomorphisms of V_τ. Let $\mathscr{K}$ and $\mathscr{A}_\mathfrak{p}$ be the universal enveloping algebras of $\mathfrak{k}_\mathbf{c}$ and $\mathfrak{a}_{\mathfrak{p}\mathbf{c}}$ respectively.

Let $\overline{\mathscr{B}}=\mathscr{B}\otimes\mathscr{E}$. $\mathscr{B}$ acts on $C^\infty(G)$ by left invariant differentiation, so there is a natural action of $\overline{\mathscr{B}}$ on $C^\infty(G)\otimes V_\tau$.

Fix h in $A_\mathfrak{p}$ such that h is not equal to the identity. Define

$$\bar{\mathfrak{k}}_2=\{X\otimes 1-1\otimes\tau_2(X)\colon X\in\mathfrak{k}_\mathbf{c}\},$$
$$\bar{\mathfrak{k}}_1=\{\mathrm{Ad}(h^{-1})X\otimes 1-1\otimes\tau_1(X)\colon X\in\mathfrak{k}_\mathbf{c}\}.$$

$\bar{\mathfrak{k}}_1$ and $\bar{\mathfrak{k}}_2$ are vector subspaces of $\overline{\mathscr{B}}$.

The following lemma generalizes [**3(g)**, Lemma 21, Corollary 2].

LEMMA 36. *$\overline{\mathscr{B}}$ is the direct sum of $\bar{\mathfrak{k}}_1\overline{\mathscr{B}}+\overline{\mathscr{B}}\bar{\mathfrak{k}}_2$ with $\mathfrak{A}_\mathfrak{p}\otimes\mathscr{E}$.*

PROOF. If ρ denotes conjugation of $\mathfrak{g}_\mathbf{c}$ with respect to the compact real form $\mathfrak{u}=\mathfrak{k}+i\mathfrak{p}$, then the inner product

$$-B(X,\rho Y),\qquad X,Y\in\mathfrak{g}_\mathbf{c}$$

converts $\mathfrak{g}_\mathbf{c}$ into a Hilbert space. Let $\mathfrak{q}$ be the orthogonal complement in $\mathfrak{g}_\mathbf{c}$ of $\mathfrak{m}_\mathbf{c}+\mathfrak{a}_{\mathfrak{p}\mathbf{c}}$ with respect to this inner product. Let $\mathfrak{q}_\mathfrak{k}=\mathfrak{q}\cap\mathfrak{k}_\mathbf{c}$, and let $S(\mathfrak{q}_\mathfrak{k})$ be the symmetric algebra on $\mathfrak{q}_\mathfrak{k}$. Let $\mathfrak{Q}_\mathfrak{k}$ be the image of $S(\mathfrak{q}\mathfrak{k})$ in $\mathscr{B}$ under the canonical mapping. (The canonical mapping is defined in [**3(a)**, p. 192]). It is proved in [**3(g)**, Lemma 21, Corollary 1] that every element in $\mathscr{B}$ has a unique representation as an element of the form

$$\sum_i q_ia_ik_i,\qquad q_i\in\mathfrak{Q}_\mathfrak{k}^{h^{-1}},\ a_i\in\mathfrak{A}_\mathfrak{p},\ k_i\in\mathscr{K}.$$

(The map $b\to b^h$, for b in $\mathscr{B}$, is the automorphism of $\mathscr{B}$ which coincides with $\mathrm{Ad}(h)$ on $\mathfrak{g}$.) We shall then write $\mathscr{B}=\mathfrak{Q}_\mathfrak{k}^{h^{-1}}\mathfrak{A}_\mathfrak{p}\mathscr{K}$. Therefore

$$\overline{\mathscr{B}}=\mathfrak{Q}_\mathfrak{k}^{h^{-1}}\mathfrak{A}_\mathfrak{p}\mathscr{K}\otimes\mathscr{E}.$$

Let $\bar{\mathfrak{q}}_{\mathfrak{k}} = \{\mathrm{Ad}(h^{-1})X \otimes 1 - 1 \otimes \tau_1(X) \colon X \in \mathfrak{q}_{\mathfrak{k}}\}$. Then it is easy to see that $\overline{\mathscr{B}}$ is the direct sum of $\bar{\mathfrak{q}}_{\mathfrak{k}}\overline{\mathscr{B}} + \overline{\mathscr{B}}\mathfrak{k}_2$ with $\mathfrak{A}_{\mathfrak{p}} \otimes \mathscr{E}$. However, in [**3(g)**, Lemma 21, Corollary 2] it is shown that

$$\mathfrak{k}^{h^{-1}}\mathscr{B} + \mathscr{B}\mathfrak{k} = \mathfrak{q}_{\mathfrak{k}}^{h^{-1}}\mathscr{B} + \mathscr{B}\mathfrak{k}.$$

Therefore

$$(\mathfrak{k}^{h^{-1}}\mathscr{B} + \mathscr{B}\mathfrak{k}) \otimes \mathscr{E} = (\mathfrak{q}_{\mathfrak{k}}^{h^{-1}}\mathscr{B} + \mathscr{B}\mathfrak{k}) \otimes \mathscr{E}.$$

It then follows that

$$\bar{\mathfrak{q}}_{\mathfrak{k}}\overline{\mathscr{B}} + \overline{\mathscr{B}}\mathfrak{k}_2 = \bar{\mathfrak{k}}_1\overline{\mathscr{B}} + \overline{\mathscr{B}}\mathfrak{k}_2.$$

This proves the lemma. □

COROLLARY. *For any b in $\mathscr{B}$ there is a unique element $\delta_h'(b)$ in $\mathfrak{A}_{\mathfrak{p}} \otimes \mathscr{E}$ such that $b - \delta_h'(b)$ is in the vector space $\bar{\mathfrak{k}}_1\overline{\mathscr{B}} + \overline{\mathscr{B}}\mathfrak{k}_2$.* □

LEMMA 37. *Let $\phi\colon G \to V_\tau$ be an infinitely differentiable τ-spherical function. Then for any b in $\overline{\mathscr{B}}$ and any h in $A_{\mathfrak{p}}$, h not equal to the identity, we have the formula*

$$\phi(h; b) = \phi(h; \delta_h'(b)).$$

(*It is clear what the notation $\phi(h; \delta_h'(b))$ means.*)

PROOF. If b_2 is in $\overline{\mathscr{B}}\mathfrak{k}_2$, then it is clear that $\phi(h; b_2) = 0$. If b_1 is in $\bar{\mathfrak{k}}_1\overline{\mathscr{B}}$, we shall assume that

$$b_1 = (\mathrm{Ad}(h^{-1})X \otimes 1 - 1 \otimes \tau_1(X))B_1$$

for some X in $\mathfrak{k}$ and some B_1 in $\overline{\mathfrak{B}}$. Then

$$\phi(h; b_1) = \phi(X; h; B_1) - \tau_1(X)\phi(h; B_1).$$

This last term is equal to zero, since $\phi(kh) = \tau_1(k)\phi(h)$ for any k in K. This is enough to prove the lemma. □

If g is in $\mathscr{B}$, identify g with the element $g \otimes 1$ in $\overline{\mathscr{B}}$. If T is in $\mathscr{E}$, identify T with the element $1 \otimes T$ in $\overline{\mathscr{B}}$.

If λ is a real linear functional on the vector space $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$, define an element H_λ in $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$ by the property

$$B(H_\lambda, H) = \lambda(H), \qquad H \in (-1)^{1/2}\mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}.$$

Also, write $\bar{\lambda}$ for that linear functional which equals λ on $\mathfrak{a}_{\mathfrak{p}}$ and equals zero on $(-1)^{1/2}\mathfrak{a}_{\mathfrak{k}}$.

Define the vectors $H_1, \ldots, H_n$, X_α, $X_{-\alpha}$ in $\mathfrak{g}_c$ and the Casimir operator $\omega_{\mathfrak{g}}$ as in §5. Since θ is an automorphism of $\mathfrak{g}_c$, $\theta(\omega_{\mathfrak{g}})$ equals $\omega_{\mathfrak{g}}$.

Fix a nonzero H in $\mathfrak{a}_{\mathfrak{p}}$ and let $h = \exp H$. We would like to compute $\delta_h'(\omega_{\mathfrak{g}})$. The following lemma will express $\delta_h'(\omega_{\mathfrak{g}})$ as a linear second-order differentiable operator on $A_{\mathfrak{p}}'$. The coefficients of this operator will be selfadjoint operators in $\mathscr{E}$ which depend on h. The lemma is a generalization of [**3(g)**, Lemma 27].

LEMMA 38. *There is a selfadjoint operator $Q(H)$ in $\mathscr{E}$ such that*

$$\delta'_h(\omega_{\mathfrak{g}}) = H_1^2 + \sum_{\alpha \in P_+} \coth \alpha(H) \cdot H_{\bar{\alpha}} + Q(H) \cdot I.$$

PROOF. $\omega_{\mathfrak{g}}$ equals $\frac{1}{2}(\omega_{\mathfrak{g}} + \theta(\omega_{\mathfrak{g}}))$. This is then equal to the expression

$$H_1^2 + \cdots + H_n^2 + \frac{1}{2} \sum_{\alpha \in P} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) + \frac{1}{2} \sum_{\alpha \in P} \theta(X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha).$$

If A and B are in $\overline{\mathscr{B}}$ we shall say A is congruent to B and write $A \equiv B$ if $A - B$ is in the vector space $\overline{\mathscr{B}}\mathfrak{k}_2 + \bar{\mathfrak{k}}_1\overline{\mathscr{B}}$. If i is greater than 1, H_i is in $(-1)^{1/2}\mathfrak{k}$ so that $H_i^2 \equiv \tau_2(H_i)^2$. This implies that $\tau_2(H_i)$ is a selfadjoint operator on V_τ, since τ_2 is a unitary representation. For any α in P let R_α be the expression

$$\tfrac{1}{2}(X_\alpha X_{-\alpha} + \theta(X_{-\alpha} X_\alpha)) + \tfrac{1}{2}(X_{-\alpha} X_\alpha + \theta(X_\alpha X_{-\alpha})).$$

Fix α such that either α or $-\alpha$ is in P_+. Let $X_\alpha = Y_\alpha + Z_\alpha$, where Y_α equals $\frac{1}{2}(X_\alpha - \theta(X_\alpha))$, a vector in $\mathfrak{p}_{\mathbf{c}}$, and Z_α equals $\frac{1}{2}(X_\alpha + \theta(X_\alpha))$, a vector in $\mathfrak{k}_{\mathbf{c}}$. Then

$$\begin{aligned} Z_\alpha^{h^{-1}} &= \tfrac{1}{2}(X_\alpha^{h^{-1}} + \theta(X_\alpha^h)) = X_\alpha(\tfrac{1}{2}e^{-\alpha(H)}) + \theta(X_\alpha)(\tfrac{1}{2}e^{\alpha(H)}) \\ &= \tfrac{1}{2}Y_\alpha(e^{-\alpha(H)} - e^{\alpha(H)}) + \tfrac{1}{2}X_\alpha(e^{-\alpha(H)} + e^{\alpha(H)}). \end{aligned}$$

Therefore we have the formula

$$Y_\alpha = \coth \alpha(H) \cdot Z_\alpha - \operatorname{csch} \alpha(H) \cdot Z_\alpha^{h^{-1}}. \tag{13.1}$$

It follows that

$$\begin{aligned} X_\alpha X_{-\alpha} &= (Y_\alpha + Z_\alpha)(Y_{-\alpha} + Z_{-\alpha}) \\ &= [(1 + \coth \alpha(H))Z_\alpha - \operatorname{csch} \alpha(H) \cdot Z_\alpha^{h^{-1}}][Y_{-\alpha} + Z_{-\alpha}] \\ &\equiv [(1 + \coth \alpha(H))Z_\alpha - \operatorname{csch} \alpha(H) \cdot \tau_1(Z_\alpha)][Y_{-\alpha} + \tau_2(Z_{-\alpha})] \\ &= (1 + \coth \alpha(H))Z_\alpha Y_{-\alpha} + (1 + \coth \alpha(H))Z_\alpha \cdot \tau_2(Z_{-\alpha}) \\ &\quad - \operatorname{csch} \alpha(H) \cdot Y_{-\alpha} \cdot \tau_1(Z_\alpha) - \operatorname{csch} \alpha(H) \cdot \tau_1(Z_\alpha)\tau_2(Z_{-\alpha}) \\ &= (1 + \coth \alpha(H))[Z_\alpha, Y_{-\alpha}] + (1 + \coth \alpha(H))Y_{-\alpha} \cdot \tau_2(Z_\alpha) \\ &\quad + (1 + \coth \alpha(H))\tau_2(Z_{-\alpha})\tau_2(Z_\alpha) - \operatorname{csch} \alpha(H) \cdot Y_{-\alpha} \cdot \tau_1(Z_\alpha) \\ &\quad - \operatorname{csch} \alpha(H)\tau_1(Z_\alpha)\tau_2(Z_{-\alpha}) \\ &= (1 + \coth \alpha(H))[Z_\alpha, Y_{-\alpha}] \\ &\quad + Y_{-\alpha}[(1 + \coth \alpha(H))\tau_2(Z_\alpha) - \operatorname{csch} \alpha(H) \cdot \tau_1(Z_\alpha)] \\ &\quad + (1 + \coth \alpha(H))\tau_2(Z_{-\alpha})\tau_2(Z_\alpha) - \operatorname{csch} \alpha(H) \cdot \tau_1(Z_\alpha)\tau_2(Z_{-\alpha}). \end{aligned}$$

We let the Cartan involution θ act on $\overline{\mathscr{B}}$ by making it act on $\mathscr{B}$ in the usual way and letting it act on $\mathscr{E}$ trivially. Notice that $\theta[Z_\alpha, Y_{-\alpha}] = -[Z_\alpha, Y_{-\alpha}]$, and $\theta(Y_{-\alpha}) = -Y_{-\alpha}$. If we let θ act on the above congruence, we obtain a new congruence with respect to the space

$$\theta(\overline{\mathscr{B}}\mathfrak{k}_2 + \bar{\mathfrak{k}}_1\overline{\mathscr{B}}) = \overline{\mathscr{B}}\mathfrak{k}_2 + \bar{\mathfrak{k}}'_1\overline{\mathscr{B}}$$

where

$$\bar{\mathfrak{k}}_1' = \{X^h \otimes 1 - 1 \otimes \tau_1(X) \colon X \in \mathfrak{k}_c\}.$$

It follows that

$$\begin{aligned}\theta(X_\alpha X_{-\alpha}) \equiv{}& -(1+\coth\alpha(H))[Z_\alpha, Y_{-\alpha}] \\ & - Y_{-\alpha}[(1+\coth\alpha(H))\tau_2(Z_\alpha) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_\alpha)] \\ & + (1+\coth\alpha(H))\tau_2(Z_{-\alpha})\tau_2(Z_\alpha) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_\alpha)\tau_2(Z_{-\alpha}) \\ & \operatorname{mod}(\overline{\mathscr{B}}\bar{\mathfrak{k}}_2 + \bar{\mathfrak{k}}_1'\bar{\mathfrak{k}}).\end{aligned}$$

In this congruence, replace α by $-\alpha$ and H by $-H$. Then $h = \exp H$ is replaced by h^{-1} and $\overline{\mathscr{B}}\bar{\mathfrak{k}}_2 + \bar{\mathfrak{k}}_1'\overline{\mathscr{B}}$ becomes $\overline{\mathscr{B}}\bar{\mathfrak{k}}_2 + \bar{\mathfrak{k}}_1\overline{\mathscr{B}}$. Then

$$\begin{aligned}\theta(X_{-\alpha} X_{\alpha}) \equiv{}& -(1+\coth\alpha(H))[Z_{-\alpha}, Y_{\alpha}] \\ & - Y_{\alpha}[(1+\coth\alpha(H))\tau_2(Z_{-\alpha}) - \operatorname{csch}\alpha(H)\tau_1(Z_{-\alpha})] \\ & + (1+\coth\alpha(H))\tau_2(Z_{\alpha})\tau_2(Z_{-\alpha}) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_{-\alpha})\tau_2(Z_{\alpha}).\end{aligned}$$

Now $[X_\alpha, X_{-\alpha}] = H_\alpha$. Therefore

$$\begin{aligned}[Z_\alpha, Y_{-\alpha}] - [Z_{-\alpha}, Y_\alpha] &= [Z_\alpha, Y_{-\alpha}] + [Y_\alpha, Z_{-\alpha}] \\ &= \tfrac14[X_\alpha + \theta(X_\alpha), X_{-\alpha} - \theta(X_{-\alpha})] + \tfrac14[X_\alpha - \theta(X_\alpha), X_{-\alpha} + \theta(X_{-\alpha})] \\ &= \tfrac12(H_\alpha - \theta(H_\alpha)) = H_{\bar\alpha}.\end{aligned}$$

We obtain the formula

$$\begin{aligned}X_\alpha X_{-\alpha} + \theta(X_{-\alpha}X_\alpha) \equiv{}& (1+\coth\alpha(H))H_\alpha \\ & + Y_{-\alpha}[(1+\coth\alpha(H))\tau_2(Z_\alpha) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_\alpha)] \\ & - Y_{\alpha}[(1+\coth\alpha(H))\tau_2(Z_{-\alpha}) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_{-\alpha})] \\ & + (1+\coth\alpha(H))[\tau_2(Z_\alpha)\tau_2(Z_{-\alpha}) + \tau_2(Z_{-\alpha})\tau_2(Z_\alpha)] \\ & - \operatorname{csch}\alpha(H)[\tau_1(Z_\alpha)\tau_2(Z_{-\alpha}) + \tau_1(Z_{-\alpha})\tau_2(Z_\alpha)].\end{aligned}$$

By substituting $-\alpha$ for α in this formula we obtain a similar expression for $X_{-\alpha}X_\alpha + \theta(X_\alpha X_{-\alpha})$. Adding the two expressions together we obtain the formula

$$\begin{aligned}R_\alpha &= \tfrac12[X_\alpha X_{-\alpha} + \theta(X_{-\alpha}X_\alpha)] + \tfrac12[X_{-\alpha}X_\alpha + \theta(X_\alpha X_{-\alpha})] \\ &\equiv \coth\alpha(H)\cdot H_\alpha + Y_{-\alpha}[\coth\alpha(H)\cdot\tau_2(Z_\alpha) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_\alpha)] \\ &\quad - Y_\alpha[\coth\alpha(H)\cdot\tau_2(Z_{-\alpha}) - \operatorname{csch}\alpha(H)\cdot\tau_1(Z_{-\alpha})] \\ &\quad + [\tau_2(Z_\alpha)\tau_2(Z_{-\alpha}) + \tau_2(Z_{-\alpha})\tau_2(Z_\alpha)].\end{aligned}$$

Using formula (13.1) for Y_α and $Y_{-\alpha}$ we find that R_α is congruent to the expression

$$\begin{aligned}\coth\alpha(H)\cdot H_{\bar\alpha} &- (\coth\alpha(H))^2[\tau_2(Z_\alpha)\tau_2(Z_{-\alpha}) + \tau_2(Z_{-\alpha})\tau_2(Z_\alpha)] \\ & - (\operatorname{csch}\alpha(H))^2[\tau_1(Z_\alpha)\tau_1(Z_{-\alpha}) + \tau_1(Z_{-\alpha})\tau_1(Z_\alpha)] \\ & + 2\operatorname{csch}\alpha(H)\coth\alpha(H)[\tau_2(Z_\alpha)\tau_1(Z_{-\alpha}) + \tau_2(Z_{-\alpha})\tau_1(Z_\alpha)] \\ & + \tau_2(Z_\alpha)\tau_2(Z_{-\alpha}) + \tau_2(Z_{-\alpha})\tau_2(Z_\alpha).\end{aligned}$$

Using the fact that $1 - (\coth\alpha(H))^2 = -(\operatorname{csch}\alpha(H))^2$ we obtain the formula

$$R_\alpha \equiv \coth\alpha(H)\cdot H_{\bar\alpha} - F_\alpha$$

where F_α is the following operator on $\mathscr{E}$:

$$(\operatorname{csch}\alpha(H))^2[\tau_2(Z_\alpha)\tau_2(Z_{-\alpha})+\tau_2(Z_{-\alpha})\tau_2(Z_\alpha) + \tau_1(Z_\alpha)\tau_1(Z_{-\alpha})+\tau_1(Z_{-\alpha})\tau_1(Z_\alpha)] - \operatorname{csch}\alpha(H)\cdot\coth\alpha(H)[\tau_2(Z_\alpha)\tau_1(Z_{-\alpha})+\tau_2(Z_{-\alpha})\tau_1(Z_\alpha)].$$

We want to show that $\sum_{\alpha\in P_+} F_\alpha$ is a selfadjoint operator on V_τ. Let η denote conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Then since τ_1 and τ_2 are unitary

$$\tau_1(X)^* = -\tau_1(\eta X), \qquad \tau_2(X)^* = -\tau_2(\eta X)$$

for any X in $\mathfrak{k}_c$. It is easy to see that there exists a constant $c_\alpha \neq 0$ such that

$$\eta X_\alpha = c_\alpha X_{\alpha^\eta}, \qquad \eta X_{-\alpha} = c_\alpha^{-1} X_{-\alpha^\eta},$$

where α^η is the conjugate root of α. Since η commutes with θ, we see that

$$\eta Z_\alpha = c_\alpha Z_{\alpha^\eta}, \qquad \eta Z_{-\alpha} = c_\alpha^{-1} Z_{-\alpha^\eta}.$$

Therefore $(F_\alpha)^* = F_{\alpha^\eta}$ for any α in P_+, since $\alpha^\eta = \alpha$ when restricted to $\mathfrak{a}_\mathfrak{p}$, and because the functions $\operatorname{csch}\alpha(H)$ and $\coth\alpha(H)$ are real-valued. Therefore

$$\left(\sum_{\alpha\in P_+} F_\alpha\right)^* = \sum_{\alpha\in P_+} F_\alpha.$$

We have shown that

$$\omega_\mathfrak{g} \equiv H_1^2 + \sum_{\alpha\in P_+} \coth\alpha(H)\cdot H_{\bar\alpha} + \sum_{i=1}^{n} \tau_2(H_i)^2 - \sum_{\alpha\in P_+} F_\alpha + \sum_{\alpha\in P_M} R_\alpha.$$

We need only show that for any α in P_M, R_α is congruent to a selfadjoint operator on V_τ. In this case

$$R_\alpha = X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha.$$

Therefore

$$R_\alpha \equiv \tau_2(X_{-\alpha})\tau_2(X_\alpha) + \tau_2(X_\alpha)\tau_2(X_{-\alpha}).$$

But $\eta R_\alpha = R_\alpha$. Therefore $\tau_2(X_{-\alpha})\tau_2(X_\alpha) + \tau_2(X_\alpha)\tau_2(X_{-\alpha})$ is a selfadjoint operator on V_τ. This completes the proof of Lemma 38. □

Recall that r_1 and r_2 stand for the number of roots in P_+ whose restrictions to $\mathfrak{a}_\mathfrak{p}$ are μ_0 and $2\mu_0$ respectively. As before, write

$$r^2 = 2(r_1 + 4r_2) = B(H_0, H).$$

Since $B(H_1,H_1) = 1$, H_1^2 equals $r^{-2}H_0^2$. Since $B(H_{\mu_0}, H_0) = \mu_0(H_0) = 1$, H_{μ_0} equals $r^{-2}H_0$. Therefore, if we write $\delta_t'(\omega_\mathfrak{g})$ for $\delta'_{\exp tH_0}(\omega_\mathfrak{g})$ and $Q(t)$ for $Q(tH_0)$, we obtain the following formula from Lemma 38:

$$r^2\delta_t'(\omega_\mathfrak{g}) = \left(\frac{d}{dt}\right)^2 + (r_1\coth t + 2r_2\coth 2t)\frac{d}{dt} + r^2 Q(t). \tag{13.2}$$

Recall that

$$D(t) = (2\sinh t)^{r_1}(2\sinh 2t)^{r_2}, \qquad t\in\mathbf{R}.$$

If t is greater than zero, $D(t)$ does not vanish. Define a new differential operator $\delta_t(\omega_{\mathfrak{g}})$ on $C^\infty(\mathbf{R}^+) \otimes V_\tau$ by

$$\delta_t(\omega_{\mathfrak{g}}) = D(t)^{1/2}\delta_t'(\omega_{\mathfrak{g}}) \circ D(t)^{-1/2}.$$

LEMMA 39. *For any $t > 0$ there is a selfadjoint operator $q(t)$ on V_τ such that*

$$r^2\delta_t(\omega_{\mathfrak{g}}) = \left(\frac{d}{dt}\right)^2 + q(t).$$

PROOF. From (13.2) we obtain the formula

$$\begin{aligned} r^2\delta_t(\omega_{\mathfrak{g}}) &= r^2 D(t)^{1/2}\delta_t'(\omega_{\mathfrak{g}}) \circ D(t)^{-1/2} \\ &= \left(\frac{d}{dt}\right)^2 + \left(r_1 \coth t + 2r_2 \coth 2t + 2D(t)^{1/2}\frac{d}{dt}D(t)^{-1/2}\right)\frac{d}{dt} + q(t). \end{aligned}$$

$q(t)$ is some operator obtained by adding real-valued scalar functions of t to the operator $Q(t)$. In particular $q(t)$ is selfadjoint. Now

$$2D(t)^{1/2} \cdot \frac{d}{dt}D(t)^{-1/2} = -D(t)^{-1} \cdot \frac{d}{dt}D(t).$$

If we differentiate $D(t)$ we obtain the formula

$$-D(t)^{-1} \cdot \frac{d}{dt}D(t) = -[r_1 \coth t + 2r_2 \coth 2t].$$

Therefore

$$r^2\delta_t(\omega_{\mathfrak{g}}) = \left(\frac{d}{dt}\right)^2 + q(t). \quad \square$$

We are now in a position to relate the linear transformations $c^+(\Lambda)$ and $c^-(\Lambda)$ of L^τ with the Plancherel measure $\beta(\sigma, \Lambda)$.

If ψ is in L^τ, $\Lambda \neq 0$, and $t \geqslant 0$, define

$$f_\Lambda(\psi\colon t) = D(t)^{1/2}E_\Lambda(\psi\colon \exp tH_0).$$

Recall that L^τ is a direct sum of a finite number of orthogonal subspaces of the form L_σ^τ. If ψ_σ is in L_σ^τ for some σ in $\mathscr{E}_M$, we have the formula

$$E_\Lambda(\psi_\sigma\colon \exp tH_0; \omega_{\mathfrak{g}}) = \pi_{\sigma,\Lambda}(\omega_{\mathfrak{g}}) \cdot E_\Lambda(\psi_\sigma\colon \exp tH_0).$$

Fix μ in L_1'. Choose σ_1 and σ_2 in $\mathscr{E}_M(\mu)$ and fix ψ_1 and ψ_2 in $L_{\sigma_1}^\tau$ and $L_{\sigma_2}^\tau$ respectively. Using Lemma 37 we see that for α equal to 1 or 2,

$$\begin{aligned} \delta_t(\omega_{\mathfrak{g}})f_\Lambda(\psi_\alpha\colon t) &= D(t)^{1/2}\delta_t'(\omega_{\mathfrak{g}})E_\Lambda(\psi_\alpha\colon \exp tH_0) \\ &= \pi_{\sigma_\alpha,\Lambda}(\omega_{\mathfrak{g}}) \cdot f_\Lambda(\psi_\alpha\colon t). \end{aligned}$$

By Lemma 39 we obtain the formula

$$\left[\left(\frac{d}{dt}\right)^2 + q(t)\right] f_\Lambda(\psi_\alpha\colon t) = r^2\pi_{\sigma_\alpha,\Lambda}(\omega_{\mathfrak{g}})f_\Lambda(\psi_\alpha\colon t), \qquad \alpha = 1, 2.$$

Write $\frac{d}{dt}f_\Lambda(\psi_\alpha : t)$ as $f'_\Lambda(\psi_\alpha : t)$. Fix $\Lambda_0 \neq 0$. Then

$$\begin{aligned}\frac{d}{dt}&[(f_\Lambda(\psi_1 : t), f'_{\Lambda_0}(\psi_2 : t)) - (f'_\Lambda(\psi_1 : t), f_{\Lambda_0}(\psi_2 : t))] \\ &= (f_\Lambda(\psi_1 : t), f''_{\Lambda_0}(\psi_2 : t)) - (f''_\Lambda(\psi_1 : t), f_{\Lambda_0}(\psi_2 : t)) \\ &= r^2(\pi_{\sigma_2,\Lambda}(\omega_{\mathfrak{g}}) - \pi_{\sigma_1,\Lambda}(\omega_{\mathfrak{g}})) \cdot (f_\Lambda(\psi_1 : t), f_{\Lambda_0}(\psi_2 : t)).\end{aligned}$$

We are using the fact that $q(t)$ is a selfadjoint operator on V_τ and that $\pi_{\sigma_2,\Lambda}(\omega_{\mathfrak{g}})$ is a real number. The scalar product above is of course on the space V_τ. Now by (12.4), $|\sigma_1| = |\sigma_2|$. Therefore, by (6.10)

$$(\pi_{\sigma_2,\Lambda}(\omega_{\mathfrak{g}}) - \pi_{\sigma_1,\Lambda}(\omega_{\mathfrak{g}})) = r^{-2}(\Lambda^2 - \Lambda_0^2).$$

For T a positive real number, define the number $V_T(\Lambda, \Lambda_0)$ by

$$V_T(\Lambda, \Lambda_0) = \int_0^T (f_\Lambda(\psi_1 : t), f_{\Lambda_0}(\psi_2 : t))\, dt. \tag{13.3}$$

Then $V_T(\Lambda, \Lambda_0)$ is equal to the expression

$$(\Lambda^2 - \Lambda_0^2)^{-1}\{(f_\Lambda(\psi_1 : t), f'_{\Lambda_0}(\psi_2 : t)) - (f'_\Lambda(\psi_1 : t), f_{\Lambda_0}(\psi_2 : t))\}|_0^T.$$

Now $D(t)^{1/2} \to 0$ as $t \to 0$, so $f_\Lambda(\psi_2 : t) \to 0$ as $t \to 0$. Therefore the evaluation of the above expression at $t = 0$ is equal to zero. We then have the formula

(13.4)
$$V_T(\Lambda, \Lambda_0) = (\Lambda^2 - \Lambda_0^2)^{-1}\{(f_\Lambda(\psi_1 : T), f'_{\Lambda_0}(\psi_2 : T)) - (f'_\Lambda(\psi_1 : T), f_{\Lambda_0}(\psi_2 : T))\}.$$

Define the integer q as in §3. Define the distribution $\Delta^T_{\Lambda_0}$on $C_0^\infty(\mathbf{R})$ by

$$\Delta^T_{\Lambda_0}(h) = \int_{-\infty}^{\infty} V_T(\Lambda, \Lambda_0) \cdot \Lambda^{2q} \cdot h(\Lambda)\, d\Lambda, \qquad h \in C_0^\infty(\mathbf{R}).$$

LEMMA 40. *If h is in $C_0^\infty(\mathbf{R})$ and $\Lambda_0 \neq 0$, the limit as T approaches infinity of $\Delta^T_{\Lambda_0}(h)$ is equal to each of the following expressions:*

(i)
$$\begin{aligned}&2\pi(c^+(\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot \Lambda_0^{2q} \cdot h(\Lambda_0) \\ &\qquad + \pi(c^-(-\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot (-\Lambda_0)^{2q} \cdot h(-\Lambda_0).\end{aligned}$$

(ii)
$$\begin{aligned}&2\pi(c^-(\Lambda_0)\psi_1, c^-(\Lambda_0)\psi_2)_M \cdot \Lambda_0^{2q} \cdot h(\Lambda_0) \\ &\qquad + \pi(c^+(-\Lambda_0)\psi_1, c^-(\Lambda_0)\psi_2)_M(-\Lambda_0)^{2q} \cdot h(-\Lambda_0).\end{aligned}$$

PROOF. Write $\theta(\psi_\alpha : t)$ as the function

$$c^+(\Lambda)\psi_\alpha(1)e^{i\Lambda t} + c^-(\Lambda)\psi_\alpha(1)e^{-i\Lambda t}.$$

Write $\frac{d}{dt}\theta_\Lambda(\psi_\alpha : t)$ as $\theta'_\Lambda(\psi_\alpha : t)$. Write $\frac{d}{dt}E_\Lambda(\psi_\alpha : \exp tH_0)$ as $E'_\Lambda(\psi_\alpha : \exp tH_0)$. Let

$$\begin{aligned}q_\alpha(\Lambda, t) &= f_\Lambda(\psi_\alpha : t) - \theta_\Lambda(\psi_\alpha : t), \\ q'_\alpha(\Lambda, t) &= f'_\Lambda(\psi_\alpha : t) - \theta'_\Lambda(\psi_\alpha : t)\end{aligned}$$

for $\alpha = 1, 2$, $\Lambda \neq 0$, and $t \geqslant 0$.

It is clear that we can find a number $\delta_1 > 0$ and a constant C such that for any $t \geqslant 0$

$$|e^{-t\rho(H_0)}D(t)^{1/2} - 1| + \left|e^{-t\rho(H_0)} \cdot \frac{d}{dt}(D(t)^{1/2}) - \rho(H_0)\right| \leqslant Ce^{-\delta_1 t}.$$

It is also clear that

$$\begin{aligned}
&\left|f'_\Lambda(\psi_\alpha\colon t) - \frac{d}{dt}[e^{t\rho(H_0)}E_\Lambda(\psi_\alpha\colon \exp tH_0)]\right| \\
&\quad \leqslant |D(t)^{1/2} \cdot E'_\Lambda(\psi_\alpha\colon \exp tH_0) - e^{t\rho(H_0)}E'_\Lambda(\psi_\alpha\colon \exp tH_0)| \\
&\qquad + \left|\frac{d}{dt}(D(t))^{1/2} \cdot E_\Lambda(\psi_\alpha\colon \exp tH_0) - \frac{d}{dt}(e^{t\rho(H_0)}) \cdot E_\Lambda(\psi_\alpha\colon \exp tH_0)\right| \\
&\quad = |e^{t\rho(H_0)}E'_\Lambda(\psi_\alpha\colon \exp tH_0)| \cdot |e^{-t\rho(H_0)}D(t)^{1/2} - 1| \\
&\qquad + |e^{t\rho(H_0)}E_\Lambda(\psi_\alpha\colon \exp tH_0)| \cdot \left|e^{-t\rho(H_0)}\frac{d}{dt}(D(t)^{1/2}) - \rho(H_0)\right|.
\end{aligned}$$

Now by (4.1) and Lemma 30, we observe that both $|e^{t\rho(H_0)}E_\Lambda(\psi_\alpha\colon \exp tH_0)|$ and $|e^{t\rho(H_0)}E'_\Lambda(\psi_\alpha\colon \exp tH_0)|$ are bounded by the product of $(1+rt)^d$ and a polynomial in $|\Lambda|$. Therefore, there exists a $\delta > 0$ and a polynomial p such that for Λ in $\mathbf{R}$ and $t \geqslant 0$

$$\left|f'_\Lambda(\psi_\alpha\colon t) - \frac{d}{dt}(e^{t\rho(H_0)}E_\Lambda(\psi_\alpha\colon \exp tH_0))\right| \leqslant p(|\Lambda|)e^{-\delta t}. \tag{13.5}$$

It then follows from (11.2) that there exists a polynomial p and an $\varepsilon > 0$ such that for $\alpha = 1, 2$, $\Lambda \neq 0$, and $t \geqslant 0$

$$|q'_\alpha(\Lambda, t)| \leqslant p(|\Lambda|)e^{-\varepsilon t}. \tag{13.6}$$

By the same argument we obtain the inequality

$$|q_\alpha(\Lambda, t)| \leqslant p(|\Lambda|)e^{-\varepsilon t}. \tag{13.7}$$

For $\Lambda \neq 0$, and $T \geqslant 0$, let $Q(\Lambda, T)$ be the sum of the following three terms:
(i) $(q_1(\Lambda, T), q'_2(\Lambda_0, T)) - (q'_1(\Lambda, T), q_2(\Lambda_0, T))$,
(ii) $(q_1(\Lambda, T), \theta'_{\Lambda_0}(\psi_2\colon T)) - (q'_1(\Lambda, T), \theta_{\Lambda_0}(\psi_2\colon T))$,
(iii) $(\theta_\Lambda(\psi_1\colon T), q'_2(\Lambda_0, T)) - (\theta'_\Lambda(\psi_1\colon T), q_2(\Lambda_0, T))$.
By Lemma 34 it is clear that for $\alpha = 1, 2$, both $|\Lambda^{2q}\theta_\Lambda(\psi_\alpha\colon T)|$ and $|\Lambda^{2q}\theta'_\Lambda(\psi_\alpha\colon T)|$ are bounded by a polynomial in $|\Lambda|$, independently of T. Therefore, using (13.6) and (13.7), we see that there is a polynomial p and an $\varepsilon > 0$ such that for Λ in $\mathbf{R}$ and $T \geqslant 0$

$$|\Lambda^{2q}Q(\Lambda, T)| \leqslant p(|\Lambda|)e^{-\varepsilon t}. \tag{13.8}$$

We shall also need a weak bound on $|\frac{d}{d\Lambda}(\Lambda^{2q}Q(\Lambda, T))|$. By using Lemma 30 and Lemma 35 we can show that there exists a polynomial p and an integer n such that for Λ in $\mathbf{R}$ and $T \geqslant 0$

$$\left|\frac{d}{d\Lambda}(\Lambda^{2q}Q(\Lambda, T))\right| \leqslant p(|\Lambda|)(1+T)^n. \tag{13.9}$$

We shall now break $V_T(\Lambda, \Lambda_0)$ up into a sum of terms which we can handle separately with the above estimates.

$$\begin{aligned}
\Lambda^{2q} V_T(\Lambda, \Lambda_0) &\\
&= \Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} \cdot \{(f_1(\psi_1 : T), f'_{\Lambda_0}(\psi_2 : T)) - (f'_\Lambda(\psi_1 : T), f_{\Lambda_0}(\psi_2 : T))\}\\
&= \Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} (\theta_\Lambda(\psi_1 : T), \theta'_{\Lambda_0}(\psi_2 : T))\\
&\quad - \Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} (\theta'_\Lambda(\psi_1 : T), \theta_{\Lambda_0}(\psi_2 : T))\\
&\quad + \Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} \cdot Q(\Lambda, T).
\end{aligned}$$

Recall that for any two vectors ψ and ψ' in L^τ

$$(\psi, \psi')_M = (\psi(1), \psi(1)).$$

Then we obtain the following formula:

$$\begin{aligned}
\Lambda^{2q} V_T(\Lambda, \Lambda_0) &\\
= i\Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} \{ & (c^+(\Lambda)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot e^{iT(\Lambda - \Lambda_0)}(\Lambda_0 + \Lambda)\\
&+ (c^-(\Lambda)\psi_1, c^-(\Lambda_0)\psi_2)_M \cdot e^{iT(-\Lambda + \Lambda_0)}(-\Lambda_0 - \Lambda)\\
&+ (c^+(\Lambda)\psi_1, c^-(\Lambda_0)\psi_2)_M \cdot e^{iT(\Lambda + \Lambda_0)}(-\Lambda_0 + \Lambda)\\
&+ (c^-(\Lambda)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot e^{iT(-\Lambda - \Lambda_0)}(\Lambda_0 - \Lambda)\}\\
+ \Lambda^{2q} \cdot (\Lambda^2 - \Lambda_0^2)^{-1} \cdot Q(\Lambda, T). &
\end{aligned}$$

Now $(\Lambda - \Lambda_0) \cdot \Lambda^{2q} \cdot V_T(\Lambda, \Lambda_0) = 0$ if $\Lambda = \Lambda_0$. Therefore, for any $T \geqslant 0$,

$$\begin{aligned}
- i\Lambda_0^{2q}\{(c^+(\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M - (c^-(\Lambda_0)\psi_1, c^-(\Lambda_0)\psi_2)_M\}&\\
+ \Lambda_0^{2q} \cdot (2\Lambda_0)^{-1} \cdot Q(\Lambda_0, T) &= 0.
\end{aligned}$$

Now by (13.8), $Q(\Lambda_0, T)$ approaches 0 as T approaches ∞, so

$$(c^+(\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M = (c^-(\Lambda_0)\psi_1, c^-(\Lambda_0)\psi_2)_M.$$

This implies that $Q(\Lambda_0, T) = 0$ for any $T \geqslant 0$. In a similar manner, we can show that

$$(c^-(-\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M = (c^+(-\Lambda_0)\psi_1, c^-(\Lambda_0)\psi_2)_M$$

and that $Q(-\Lambda_0, T) = 0$ for any $T \geqslant 0$.

The number

$$\Delta^T_{\Lambda_0}(h) = \int_{\mathbf{R}} h(\Lambda) \cdot \Lambda^{2q} \cdot V_T(\Lambda, \Lambda_0)\, d\Lambda$$

is equal to the sum of the following three terms:

(i)

$$\begin{aligned}
&\int_{\mathbf{R}} \{(c^+(\Lambda)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot e^{iT(\Lambda - \Lambda_0)}\\
&\qquad - (c^-(\Lambda)\psi_1, c^-(\Lambda_0)\psi_2)_M \cdot e^{-iT(\Lambda - \Lambda_0)}\} \cdot h(\Lambda)(-i)\Lambda^{2q} \cdot (\Lambda - \Lambda_0)^{-1}\, d\Lambda,
\end{aligned}$$

(ii)

$$\begin{aligned}
&\int_{\mathbf{R}} \{(c^+(\Lambda)\psi_1, c^-(\Lambda_0)\psi_2)_M \cdot e^{iT(\Lambda + \Lambda_0)}\\
&\qquad - (c^-(\Lambda)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot e^{-iT(\Lambda + \Lambda_0)}\} \cdot h(\Lambda)(-i)\Lambda^{2q} \cdot (\Lambda + \Lambda_0)^{-1}\, d\Lambda,
\end{aligned}$$

(iii)
$$\int_{\mathbf{R}} Q(\Lambda,T)\cdot\Lambda^{2q}\cdot h(\Lambda)\cdot(\Lambda^2-\Lambda_0^2)^{-1}\,d\Lambda.$$

Let us assume for the moment that h vanishes in a neighborhood of $(-\Lambda_0)$. We wish to examine the behavior of each of the above three terms as T approaches ∞. We deal first with the third term.

Let n be the integer in (13.9). The third term is equal to I_1+I_2 where I_1 and I_2 are those parts of the integral in (iii) which are taken over the sets $\{\Lambda\colon |\Lambda-\Lambda_0|\leqslant T^{-n-1}\}$ and $\{\Lambda\colon |\Lambda-\Lambda_0|>T^{-n-1}\}$ respectively. Apply the mean value theorem to the function $\Lambda^{2q}Q(\Lambda,T)$ of Λ. Since $Q(\Lambda_0,T)=0$, there is a real Λ_1, with $|\Lambda_1-\Lambda_0|<|\Lambda-\Lambda_0|$, such that
$$\Lambda^{2q}Q(\Lambda,T)=(\Lambda-\Lambda_0)\cdot\frac{d}{d\Lambda_1}(\Lambda_1^{2q}Q(\Lambda_1,T)).$$
Then by (13.9) we see that $|I_1|$ is bounded by a constant multiple of $(1+T^n)T^{-n-1}$. Therefore, $|I_1|$ approaches 0 as $T\to\infty$. On the other hand, there is a constant C, independent of Λ and T, such that if $|\Lambda-\Lambda_0|>T^{-n-1}$,
$$|\Lambda^{2q}\cdot h(\Lambda)\cdot(\Lambda^2-\Lambda_0^2)^{-1}\cdot Q(\Lambda,T)|\leqslant Cp(|\Lambda|)\cdot|h(\Lambda)|\cdot T^{n+1}\cdot e^{-\varepsilon t}$$
by (13.8). (We have used the fact that $h(\Lambda)$ vanishes in a neighborhood of $-\Lambda_0$.) This implies that $|I_2|$ approaches 0 as T approaches ∞. Therefore, term (iii) goes to 0 as T goes to ∞.

To deal with term (ii), we observe that both
$$(-i)\cdot\Lambda^{2q}\cdot(\Lambda+\Lambda_0)^{-1}\cdot(c^+(\Lambda)\psi_1,c^-(\Lambda_0)\psi_2)_M\cdot h(\Lambda)$$
and
$$(-i)\cdot\Lambda^{2q}\cdot(\Lambda+\Lambda_0)^{-1}\cdot(c^-(\Lambda)\psi_1,c^+(\Lambda_0)\psi_2)_M\cdot h(\Lambda)$$
are continuous functions of Λ, since $h(\Lambda)$ vanishes in a neighborhood of $(-\Lambda_0)$. Therefore, by the Riemann-Lebesgue lemma, term (ii) goes to 0 as T goes to ∞.

Now, since $(c^+(\Lambda_0)\psi_1,c^+(\Lambda_0)\psi_2)_M = (c^-(\Lambda_0)\psi_1,c^-(\Lambda_0)\psi_2)_M$, we may rewrite term (i) as
$$\begin{aligned}
2&\int_{\mathbf{R}}(c^+(\Lambda_0)\psi_1,c^+(\Lambda_0)\psi_2)_M\cdot\Lambda^{2q}\cdot h(\Lambda)\cdot\sin T(\Lambda-\Lambda_0)\cdot(\Lambda-\Lambda_0)^{-1}\,d\Lambda\\
&+(-i)\int_{\mathbf{R}}(c^+(\Lambda)\psi_1-c^+(\Lambda_0)\psi_1,c^+(\Lambda_0)\psi_2)_M\cdot(\Lambda-\Lambda_0)^{-1}\cdot\Lambda^{2q}\cdot h(\Lambda)\\
&\qquad\cdot e^{iT(\Lambda-\Lambda_0)}\,d\Lambda\\
&+(i)\int_{\mathbf{R}}(c^-(\Lambda)\psi_1-c^-(\Lambda_0)\psi_1,c^-(\Lambda_0)\psi_2)_M\cdot(\Lambda-\Lambda_0)^{-1}\cdot\Lambda^{2q}\cdot h(\Lambda)\\
&\qquad\cdot e^{-iT(\Lambda-\Lambda_0)}\,d\Lambda\\
&=J_1+J_2+J_3,\quad\text{say.}
\end{aligned}$$

Now $\Lambda^{2q}c^+(\Lambda)\psi_1$ and $\Lambda^{2q}c^-(\Lambda)\psi_1$ are both continuously differentiable in Λ by Lemma 35. Therefore, we can use the Riemann-Lebesgue lemma to prove that
$$\lim_{T\to\infty}(|J_2|+|J_3|)=0.$$

It follows that

$$\lim_{T\to\infty}\Delta^T_{\Lambda_0}(h)$$

$$= \lim_{T\to\infty} 2\int_{\mathbf{R}} (c^+(\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M \cdot \Lambda^{2q}\cdot h(\Lambda)\cdot \sin T(\Lambda-\Lambda_0) \cdot (\Lambda-\Lambda_0)^{-1}\, d\Lambda.$$

Now it is an easy fact that

$$\lim_{T\to\infty} \sin T(\Lambda-\Lambda_0)\cdot(\Lambda-\Lambda_0)^{-1} = \pi\delta_{\Lambda_0}.$$

where the limit is taken in the topology of $\mathscr{S}'(\mathbf{R})$ and δ_{Λ_0} is the Dirac distribution at Λ_0.

We have proved that if h vanishes in a neighborhood of $(-\Lambda_0)$,

$$\lim_{T\to\infty}\Delta^T_{\Lambda_0}(h) = 2\pi(c^+(\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M\cdot \Lambda_0^{2q}\cdot h(\Lambda_0).$$

Similarly, if h vanishes in a neighborhood Λ_0, we could show that

$$\lim_{T\to\infty}\Delta^T_{\Lambda_0}(h) = 2\pi(c^-(-\Lambda_0)\psi_1, c^+(\Lambda_0)\psi_2)_M\cdot (-\Lambda_0)^{2q}\cdot h(-\Lambda_0).$$

This completes the proof of Lemma 40. ◻

For any σ in $\mathscr{E}_M$ and Λ in $\mathbf{R}$, $\beta(\sigma,\Lambda)=\beta(\sigma',\Lambda)$ by Lemma 5. Then if σ_1 and σ_2 are in $\mathscr{E}_M(\mu)$ for some μ in L_1', $\beta(\sigma_1,\Lambda)=\beta(\sigma_2,\Lambda)$. We write this number as $\beta(\mu,\Lambda)$. It is also easy to see from the Weyl dimension formula and the definition of $\mathscr{E}_M(\mu)$ that if σ_1 and σ_2 are in $\mathscr{E}_M(\mu)$, $\dim\sigma_1=\dim\sigma_2$. We write this number as $\dim\mu$.

LEMMA 41. *Assume $\Lambda_0\neq 0$. Fix μ in L_1'. Then there is a constant C, independent of Λ_0, such that the following operator equations hold on L^τ_μ:*

(i) $c^+(\Lambda_0)^*c^+(\Lambda_0) = c^-(\Lambda_0)^*c^-(\Lambda_0) = (C\cdot\beta(\mu,\Lambda_0))^{-1}$,

(ii) $c^+(\Lambda_0)^*c^-(-\Lambda_0) = c^-(\Lambda_0)^*c^+(-\Lambda_0) = (C\cdot\beta(\mu,\Lambda_0))^{-1}\cdot M(-\Lambda_0)$.

PROOF. In §12 we remarked that L_μ was an invariant subspace of $c^\pm(\Lambda_0)$. Then L_μ is also invariant under $c^\pm(\Lambda_0)^*$. Then from Lemma 40 we obtain the equations

$$c^+(\Lambda_0)^*c^+(\Lambda_0) = c^-(\Lambda_0)^*c^-(\Lambda_0),$$
$$c^+(\Lambda_0)^*c^-(-\Lambda_0) = c^-(\Lambda_0)^*c^+(-\Lambda_0).$$

Now fix ψ_1 in $L^\tau_{\sigma_1}$ and ψ_2 in $L^\tau_{\sigma_2}$ for σ_1, σ_2 in $\mathscr{E}_M(\mu)$. Also fix h in $C_0^\infty(\mathbf{R})$. Then

$$\lim_{T\to\infty}\Delta^T_{\Lambda_0}(h) = \lim_{T\to\infty}\int_{\mathbf{R}} h(\Lambda)\cdot\Lambda^{2q}\cdot V_T(\Lambda,\Lambda_0)\,d\Lambda$$

$$= \lim_{T\to\infty}\int_0^T\int_{\mathbf{R}} (f_\Lambda(\psi_1\colon t), f_{\Lambda_0}(\psi_2\colon t))\cdot\Lambda^{2q}\cdot h(\Lambda)\,d\Lambda\,dt$$

$$= \lim_{T\to\infty}\int_0^T D(t)\int_{\mathbf{R}} (E_\Lambda(\psi_1\colon \exp tH_0), E_{\Lambda_0}(\psi_2\colon \exp tH_0)) \cdot\Lambda^{2q}\cdot h(\Lambda)\,d\Lambda\,dt$$

$$= \lim_{T\to\infty}(c)^{-1}\int_{G_T}\int_{\mathbf{R}} (E_\Lambda(\psi_1\colon x), E_{\Lambda_0}(\psi_2\colon x))\cdot\Lambda^{2q}\cdot h(\Lambda)\,d\Lambda\,dx$$

where $G_T = \{k_1 \cdot \exp tH_0 \cdot k_2\colon k_1, k_2 \in K,\ 0 \leqslant t \leqslant T\}$. Let us assume that h vanishes in a neighborhood of 0. Then we define

$$u(\Lambda) = \Lambda^{2q} \cdot h(\Lambda) \cdot \beta(\mu, \Lambda)^{-1}\psi_1, \qquad \Lambda \in \mathbf{R}.$$

$\beta(\mu, \Lambda)$ is infinitely differentiable and does not vanish if $\Lambda \neq 0$ by Lemma 5. Therefore $u \in C_0^\infty(\mathbf{R}) \otimes L^\tau$.

Then in the notation of (7.4) and (7.5)

$$\lim_{T\to\infty} \Delta^T_{\Lambda_0}(h) = (c \cdot \dim \mu)^{-1}(\hat{u}(\Lambda_0), \psi_2)_M.$$

By (7.6) the right-hand side of this formula equals

$$(2c \cdot \dim \mu)^{-1}(u(\Lambda_0) + M(-\Lambda_0)u(-\Lambda_0), \psi_2)_M,$$

which in turn is equal to

$$\begin{aligned}(2c \cdot \dim \mu \cdot \beta(\mu, \Lambda_0))^{-1}\Lambda_0^{2q} \cdot h(\Lambda_0) \cdot (\psi_1, \psi_2)_M \\ + (2c \cdot \dim \mu \cdot \beta(\mu, \Lambda_0))^{-1}\Lambda_0^{2q} \cdot h(\Lambda_0) \cdot (M(-\Lambda_0)\psi_1, \psi_2)_M.\end{aligned}$$

However, by the last lemma, $\lim_{T\to\infty} \Delta^T_{\Lambda_0}(h)$ equals

$$\begin{aligned}2\pi \cdot \Lambda_0^{2q} \cdot h(\Lambda_0) \cdot (c^+(\Lambda_0)^* c^+(\Lambda_0)\psi_1, \psi_2)_M \\ + 2\pi \cdot (-\Lambda_0)^{2q} \cdot h(-\Lambda_0) \cdot (c^+(\Lambda_0)^* c^-(-\Lambda_0)\psi_1, \psi_2)_M.\end{aligned}$$

We can choose h so that $h(\Lambda_0)$ and $h(-\Lambda_0)$ are arbitrary. This proves the formulae

$$\begin{aligned}c^+(\Lambda_0)^* c^+(\Lambda_0) &= (\pi \cdot c \cdot \dim \mu \cdot \beta(\mu, \Lambda_0))^{-1}, \\ c^+(\Lambda_0)^* c^-(-\Lambda_0) &= (\pi \cdot c \cdot \dim \mu \cdot \beta(\mu, \Lambda_0))^{-1} \cdot M(-\Lambda_0). \quad \square\end{aligned}$$

14. A condition for irreducibility of $\pi_{\sigma,0}$. We have now done enough work to prove Theorem 3′(b). However, we shall postpone this until §15. In this chapter we shall use the inequality (11.2) and Lemma 41 to give a sufficient condition for the irreducibility of the representation $\pi_{\sigma,0}$.

LEMMA 42. *For any ψ in L^τ_σ, where τ is in $\mathscr{E}^2_K$ and σ is in $\mathscr{E}_M$, the meromorphic functions $c^+(\Lambda)\psi(1)$ and $c^-(\Lambda)\psi(1)$ have a pole of order at most one at $\Lambda = 0$.*

PROOF. If ψ is fixed, then by (11.2) and (12.3) there is a polynomial p and an $\varepsilon > 0$ such that

$$\begin{aligned}&|e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0) - c^+(\Lambda)\psi(1) \cdot e^{i\Lambda t} - c^-(\Lambda)\psi(1) \cdot e^{-i\Lambda t}| \\ &\qquad \leqslant p(|\Lambda|)e^{-\varepsilon t}, \qquad t \geqslant 0,\ \Lambda \neq 0.\end{aligned} \tag{14.1}$$

Therefore, for any $t \geqslant 0$, the function

$$e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0) - c^+(\Lambda)\psi(1) \cdot e^{i\Lambda t} - c^-(\Lambda)\psi(1) \cdot e^{-i\Lambda t}$$

has no pole at $\Lambda = 0$. However, the function

$$e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0)$$

is regular at $\Lambda = 0$, so the function

$$c^+(\Lambda)\psi(1)\cdot e^{i\Lambda t} + c^-(\Lambda)\psi(1)\cdot e^{-i\Lambda t}$$

is also regular at $\Lambda = 0$.

Consider the Laurent expansions of the functions $c^+(\Lambda)\psi(1)\cdot e^{i\Lambda t}$ and $c^-(\Lambda)\psi(1)\cdot e^{-i\Lambda t}$. The coefficients will be functions of t with values in V_τ, the space on which τ acts. Allow t to vary. Suppose that one of the two functions has a pole of order greater than one at $\Lambda = 0$. Then it is easy to see from the Laurent expansions that this forces the function

$$c^+(\Lambda)\psi(1)\cdot e^{i\Lambda t} + c^-(\Lambda)\psi(1)\cdot e^{-i\Lambda t}$$

to have a pole at $\Lambda = 0$ for some $t \geqslant 0$. We have a contradiction. □

THEOREM 4. *If σ is in $\mathscr{E}_M$ and $\beta(\sigma, 0) = 0$, then the representation $\pi_{\sigma,0}$ of G is irreducible.*

PROOF. Assume the contrary. Then as we saw in §5, there is a τ in $\mathscr{E}_K^2$, acting on the space V_τ, and a nonzero ψ in L_σ^τ such that $E_0(\psi\colon \exp tH_0)$ vanishes for all t. We shall use the inequality (14.1) to obtain a contradiction.

Since $E_0(\psi\colon \exp tH_0)$ vanishes for all $t \geqslant 0$, we can use the mean value theorem to show that for every Λ, t,

$$E_\Lambda(\psi\colon \exp tH_0) = \Lambda\cdot\frac{d}{d\Lambda_1}E_{\Lambda_1}(\psi\colon \exp tH_0).$$

Λ_1 is some real number between 0 and Λ. However, by Lemma 30, there is a polynomial p_1 such that for any $t \geqslant 0$,

$$\left|e^{t\rho(H_0)}\frac{d}{d\Lambda_1}E_{\Lambda_1}(\psi\colon \exp tH_0)\right| \leqslant p_1(|\Lambda_1|)\cdot\Xi(\exp tH_0)\cdot t.$$

But by (4.1)

$$\Xi(\exp tH_0)\cdot t \leqslant e^{-t\rho(H_0)}\cdot(1+t)^{d+1}, \qquad t \geqslant 0.$$

Therefore there is a polynomial p_2 such that for $\Lambda \in \mathbf{R}$ and $t \geqslant 0$,

$$(14.2) \qquad |e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0)| \leqslant |\Lambda|\cdot p_2(|\Lambda|)\cdot(1+t)^{d+1}.$$

Since $\beta(\sigma, 0) = 0$, Lemma 41 tells us that the functions $c^+(\Lambda)\psi(1)$ and $c^-(\Lambda)\psi(1)$ must both have a pole at $\Lambda = 0$. This pole must be of order 1 by Lemma 42. Let

$$c^\pm(\Lambda)\psi(1) = c_{-1}^\pm\Lambda^{-1} + c_0^\pm + c_1^\pm\Lambda + \cdots$$

be the Laurent expansions about $\Lambda = 0$ for these functions. $c_{-1}^\pm, c_0^\pm, c_1^\pm, \ldots$ are vectors in V_τ, and neither c_{-1}^+ nor c_{-1}^- can equal zero. If $t \geqslant 0$ is fixed, we apply Taylor's formula with remainder to the functions

$$g^\pm(t,\Lambda) = \Lambda c^\pm(\Lambda)\psi(1)e^{\pm i\Lambda t}.$$

Then there is a real number Λ_1 between 0 and Λ such that

$$g^+(t,\Lambda) = c_{-1}^+ + \Lambda(c_0^+ + ic_{-1}^+t) + \Lambda^2\cdot\left(\frac{d}{d\Lambda_1}\right)^2 g^+(t,\Lambda_1).$$

The function $(d/d\Lambda_1)^2 g^+(t,\Lambda_1)$ is equal to

$$e^{i\Lambda_1 t}\left(\frac{d}{d\Lambda_1}\right)^2(\Lambda_1 c^+(\Lambda_1)\psi(1)) + e^{i\Lambda_1 t}\cdot 2it\cdot \frac{d}{d\Lambda_1}(\Lambda_1 c^+(\Lambda_1)\psi(1)) \\ - e^{i\Lambda_1 t}\cdot t^2\cdot(\Lambda_1 c^+(\Lambda_1)\psi(1)).$$

Then by Lemma 35 there exists a polynomial p_3 such that

$$\left|\left(\frac{d}{d\Lambda_1}\right)^2 g^+(t,\Lambda_1)\right| \leqslant p_3(|\Lambda_1|)(1+t)^2.$$

This implies that we can choose a polynomial p_4 such that for $\Lambda \neq 0$ and $t \geqslant 0$,

$$|c^+(\Lambda)\psi(1)e^{i\Lambda t} - (c_{-1}^+\Lambda^{-1} + c_0^+ + ic_{-1}^+ t)| \leqslant |\Lambda|\cdot p_4(|\Lambda|)\cdot(1+t)^2. \tag{14.3}$$

Similarly, we can show that

$$|c^-(\Lambda)\psi(1)e^{-i\Lambda t} - (c_{-1}^-\Lambda^{-1} + c_0^- - ic_{-1}^- t)| \leqslant |\Lambda|\cdot p_4(|\lambda|)\cdot(1+t)^2. \tag{14.4}$$

Now for $\Lambda \neq 0$ and $t \geqslant 0$ the expression

$$|(c_{-1}^+ + c_{-1}^-)\Lambda^{-1} + (c_0^+ + c_0^-) + i(c_{-1}^+ - c_{-1}^-)t|$$

is bounded by the sum of the following four terms:

(i) $|c^+(\Lambda)\psi(1)e^{i\Lambda t} - (c_{-1}^+\Lambda^{-1} + c_0^+ + ic_{-1}^+ t)|$,
(ii) $|c^-(\Lambda)\psi(1)e^{-i\Lambda t} - (c_{-1}^-\Lambda^{-1} + c_0^- - ic_{-1}^- t)|$,
(iii) $|e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0)|$,
(iv) $|e^{t\rho(H_0)}E_\Lambda(\psi\colon \exp tH_0) - c^+(\Lambda)\psi(1)e^{i\Lambda t} - c^-(\Lambda)\psi(1)e^{-i\Lambda t}|$.

By (14.1), (14.2), (14.3), and (14.4), there are polynomials p and p_5 such that this sum is bounded by

$$p(|\Lambda|)e^{-\varepsilon t} + |\Lambda|\cdot p_5(|\Lambda|)\cdot(1+t)^{d+2}.$$

For a fixed t and for $0 < \Lambda \leqslant 1$ this expression is clearly bounded. Therefore

$$c_{-1}^+ + c_{-1}^- = 0. \tag{14.5}$$

Let

$$P = \sup_{0<\Lambda\leqslant 1} p(\Lambda), \qquad P_5 = \sup_{0<\Lambda\leqslant 1} p_5(\Lambda).$$

Then for any $t \geqslant 0$ and for $0 < \Lambda \leqslant 1$,

$$|(c_0^+ + c_0^-) + i(c_{-1}^+ - c_{-1}^-)t| \leqslant P\cdot e^{-\varepsilon t} + P_5\cdot\Lambda\cdot(1+t)^{d+2}.$$

Let $\Lambda = t^{-(d+3)}$, and let t approach ∞. Then we see that

$$c_{-1}^+ - c_{-1}^- = 0.$$

Therefore, by (14.5)

$$c_{-1}^+ = c_{-1}^- = 0.$$

We have a contradiction, so Theorem 4 is proved. □

15. Completion of the proof of Theorem 3′(b).

LEMMA 43. *Suppose that ψ is in L^τ_σ for τ in $\mathscr{E}^2_K$, σ in $\mathscr{E}_M$, and assume that $\|\psi\|_M = 1$. Then $c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda)$ is an infinitely differentiable function from $\mathbf{R}$ into V_τ, the space on which τ acts. Furthermore, for every nonnegative integer n there are polynomials p_1, p_2, q independent of σ, Λ, τ such that*

$$\left|\left(\frac{d}{d\Lambda}\right)^n (c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda))\right| \leqslant p_1(|\sigma|)\cdot p_2(|\Lambda|)\cdot q(|\tau|).$$

PROOF. $c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda)$ is equal to $\Lambda^{2q}c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda)\Lambda^{-2q}$. This is the restriction to $\mathbf{R}$ of a function $h(\Lambda)$ which is meromorphic in the region $|\Lambda_I| < \delta$. (See Lemma 5, (ii), and the corollary to Lemma 34.) The only possible real pole of $h(\Lambda)$ is at $\Lambda = 0$. For any real $\Lambda \neq 0$

$$\begin{aligned}|c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda)| &= \beta(\sigma,\Lambda)\cdot(c^{\pm}(\Lambda)\psi, c^{\pm}(\Lambda)\psi)^{1/2}\\ &= (c^{\pm}(\Lambda)^* c^{\pm}(\Lambda)\psi,\psi)^{1/2}\cdot\beta(\sigma,\Lambda).\end{aligned}$$

Therefore, by Lemma 41,

$$|c^{\pm}(\Lambda)\psi(1)\cdot\beta(\sigma,\Lambda)| = (C)^{-1}\cdot(\psi,\psi)^{1/2}\cdot\beta(\sigma,\Lambda)^{1/2}. \tag{15.1}$$

But by Lemma 5, $\beta(\sigma,\Lambda)$ is regular at $\Lambda = 0$, so $|h(\Lambda)|$ is bounded for all real Λ in a neighborhood of 0. Therefore, $h(\Lambda)$ has no real pole and hence is infinitely differentiable at any real Λ.

Since $\beta(\sigma,\Lambda)$ is regular at $\Lambda = 0$, the functions $c^{\pm}(\Lambda)\psi(1)$ cannot have a zero at $\Lambda = 0$. Let $N(\psi)$ be the order of the pole of the functions $c^{+}(\Lambda)\psi(1)$ and $c^{-}(\Lambda)\psi(1)$ at $\Lambda = 0$. $N(\psi)$ is a nonnegative integer. We see from (15.1) that $\beta(\sigma,\Lambda)$ has a zero of order $2N(\psi)$ at $\Lambda = 0$. Then write

$$c^{\pm}(\Lambda)\psi(1)\beta(\sigma,\Lambda) = \Lambda^{N(\psi)}c^{\pm}(\Lambda)\psi(1)\cdot\Lambda^{-N(\psi)}\beta(\sigma,\Lambda).$$

The estimate for $(d/d\Lambda)^n(c^{\pm}(\Lambda)\psi(1)\beta(\sigma,\Lambda))$ in the lemma then follows from Lemma 5, (iv), Lemma 35, and Leibnitz' rule. ◻

LEMMA 44. *Choose ψ in L^τ_σ with $\|\psi\|_M = 1$. Then for every nonnegative integer s there are polynomials p_1, p_2, q such that for every h in $\mathscr{S}(\mathbf{R})$,*

$$\sup_{x\in G}\left|\int_{\mathbf{R}} h(\Lambda)E_\Lambda(\psi\colon x)\beta(\sigma,\Lambda)\,d\Lambda\cdot\Xi(x)^{-1}(1+\sigma(x))^s\right|$$
$$\leqslant p_1(|\sigma|)\cdot q(|\tau|)\cdot\sup_{0\leqslant i\leqslant s}\cdot\sup_{\Lambda\in\mathbf{R}}\left(p_2(|\Lambda|)\left|\left(\frac{d}{d\Lambda}\right)^i h(\Lambda)\right|\right).$$

PROOF. Every x in G is of the form $k_1\cdot\exp tH_0\cdot k_2$, for k_1, k_2 in K, and $t \geqslant 0$. By (4.1) it is enough to prove the lemma when the left-hand side of the inequality is replaced by

$$\sup_{t\geqslant 0}\left|e^{t\rho(H_0)}(1+rt)^s\int_{-\infty}^{\infty}E_\Lambda(\psi\colon \exp tH_0)\cdot h(\Lambda)\cdot\beta(\sigma,\Lambda)\,d\Lambda\right|.$$

If $t \geqslant 0$, the expression

$$\left|e^{t\rho(H_0)}(1+rt)^s\int_{-\infty}^{\infty}E_\Lambda(\psi\colon \exp tH_0)\cdot h(\Lambda)\cdot\beta(\sigma,\Lambda)\,d\Lambda\right|$$

is bounded by the sum of the following two expressions:

(i) $I_1(t) = (1+rt)^s \cdot \int_{-\infty}^{\infty} |h(\Lambda)| \cdot |\beta(\sigma, \Lambda)|$
$$\cdot |e^{t\rho(H_0)} E_\Lambda(\psi\colon \exp tH_0) - c^+(\Lambda)\psi(1)e^{i\Lambda t} - c^-(\Lambda)\psi(1)e^{-i\Lambda t}|\, d\Lambda,$$

(ii) $I_2(t) = |\int_{-\infty}^{\infty} (c^+(\Lambda)\psi(1)e^{i\Lambda t} - c^-(\Lambda)\psi(1)e^{-i\Lambda t})h(\Lambda)\beta(\sigma,\Lambda)\, d\Lambda \cdot (1+rt)^s|$.

By (11.2) and (12.3), there are polynomials p_1, p_2, q, and a number $\varepsilon > 0$, all independent of σ, Λ, τ, such that
$$I_1(t) \leqslant p_1(|\sigma|) \cdot q(|\tau|) \cdot (1+rt)^s \cdot e^{-\varepsilon t} \cdot \int_{\mathbf{R}} p_2(|\Lambda|) \cdot |h(\Lambda)||\beta(\sigma,\Lambda)|\, d\Lambda.$$
But by Lemma 5, (iv), there exist polynomials p_1' and p_2', such that
$$\beta(\sigma,\Lambda) \leqslant p_1'(|\sigma|) \cdot p_2'(|\Lambda|).$$
Therefore
$$\sup_{t\geqslant 0} I_1(t) \leqslant c_s \cdot p_1(|\sigma|) \cdot p_1'(|\sigma|) \cdot q(|\tau|)$$
$$\cdot \sup_{\Lambda\in\mathbf{R}} (p_2(|\Lambda|) \cdot p_2'(|\Lambda|) \cdot (1+|\Lambda|^2) \cdot |h(\Lambda)|),$$
where
$$c_s = \int_{\mathbf{R}} (1+|\Lambda|^2)^{-1}\, d\Lambda \cdot \sup_{t\geqslant 0}(1+rt)^s e^{-\varepsilon t}.$$
This takes care of $I_1(t)$.

We now obtain a bound for $I_2(t)$. Define
$$\phi^{\pm}(\Lambda) = c^{\pm}(\Lambda)\psi(1)\beta(\sigma,\Lambda) \cdot h(\Lambda).$$
$\phi^{\pm}$ is in $\mathscr{S}(\mathbf{R}) \otimes V_\tau$. We see that
$$I_2(t) \leqslant \left|\int_{-\infty}^{\infty} \phi^+(\Lambda)e^{i\Lambda t}\, d\Lambda\right| \cdot (1+rt)^s + \left|\int_{-\infty}^{\infty} \phi^-(\Lambda)e^{-i\Lambda t}\, d\Lambda\right| \cdot (1+rt)^s$$
$$= \left|\int_{-\infty}^{\infty} e^{i\Lambda t}\left(1 + ir \cdot \frac{d}{d\Lambda}\right)^s \phi^+(\Lambda)\, d\Lambda\right|$$
$$+ \left|\int_{-\infty}^{\infty} e^{-i\Lambda t}\left(1 - ir \cdot \frac{d}{d\Lambda}\right)^s \phi^-(\Lambda)\, d\Lambda\right|.$$
Now by Liebnitz' rule and Lemma 43 there are polynomials p_1, p_2, q such that
$$\sup_{\Lambda\in\mathbf{R}} \left|(1+|\Lambda|^2)\left(1 \pm ir \cdot \frac{d}{d\Lambda}\right)^s \phi^{\pm}(\Lambda)\right|$$
$$\leqslant p_1(|\sigma|) \cdot q(|\tau|) \cdot \sup_{1\leqslant i\leqslant s} \cdot \sup_{\Lambda\in\mathbf{R}} \left(p_2(|\Lambda|) \cdot \left|\left(\frac{d}{d\Lambda}\right)^i h(\Lambda)\right|\right).$$
Therefore, we see that
$$I_2(t) \leqslant c \cdot p_1(|\sigma|) \cdot q(|\tau|) \cdot \sup_{1\leqslant i\leqslant s} \cdot \sup_{\Lambda\in\mathbf{R}} \left(p_2(|\Lambda|) \cdot \left|\left(\frac{d}{d\Lambda}\right)^i h(\Lambda)\right|\right)$$
where $c = 2\int_{-\infty}^{\infty}(1+|\Lambda|^2)^{-1}\, d\Lambda$. We have proved Lemma 44. □

COROLLARY. *Suppose ψ is in L_σ^τ and $\|\psi\|_M = 1$. Then for every g_1, g_2 in $\mathscr{B}$, and every nonnegative integer s, there are polynomials p_1, p_2, q such that for every h in $\mathscr{S}(\mathbf{R})$,*

$$\sup_{x\in G}\left|\int_{\mathbf{R}} h(\Lambda)E_\Lambda(\psi\colon g_1; x; g_2)\cdot\beta(\sigma,\Lambda)\,d\Lambda\cdot\Xi(x)^{-1}(1+\sigma(x))^s\,dx\right|$$
$$\leqslant p_1(|\sigma|)\cdot q(|\tau|)\cdot\sup_{1\leqslant i\leqslant s}\cdot\sup_{\Lambda\in\mathbf{R}}\left(p_2(|\Lambda|)\left|\left(\frac{d}{d\Lambda}\right)^i h(\Lambda)\right|\right).$$

PROOF. By (11.1)

$$|E_\Lambda(\psi\colon g_1; x; g_2)|^2 = \sum_{\alpha=1}^{t_1}\sum_{\beta=1}^{t_2}|(\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(x)\Psi_{2\beta})|^2$$

where $\Psi_{1\alpha}$ and $\Psi_{2\beta}$ are the same as in (11.1). Now let $E_\Lambda(\psi_{\alpha,\beta}\colon x)$ be the $(\tau_{1\alpha}, \tau_{2\beta})$-spherical function corresponding to $(\Psi_{1\alpha}, \pi_{\sigma,\Lambda}(x)\Psi_{2\beta})$. Apply Lemma 44 to $E_\Lambda(\psi_{\alpha,\beta}\colon x)$. The corollary follows from the conditions on $\{\Psi_{1\alpha}\}$, $\{\Psi_{2\beta}\}$ and $\{\tau_{1\alpha}\}$, $\{\tau_{2\beta}\}$ given in Lemma 13. ◻

This corollary establishes the proof of Theorem 3′(b). The proof of Theorem 3 is now complete.

16. Tempered distributions. Having proved Theorem 3, we can now extend the definition of Fourier transform to tempered distributions on G.

A distribution on G is said to be tempered if it extends to a continuous linear functional from $\mathscr{C}(G)$ to $\mathbf{C}$. Since $C_0^\infty(G)$ is dense in $\mathscr{C}(G)$, and since the inclusion map

$$C_0^\infty(G) \subset \mathscr{C}(G)$$

is continuous, we can regard the space of tempered distributions as the dual space of $\mathscr{C}(G)$, that is, the space of continuous linear functionals from $\mathscr{C}(G)$ into $\mathbf{C}$.

Let $\mathscr{C}'(G)$ be the set of tempered distributions on G. It becomes a locally convex topological vector space when endowed with the weak topology. (A base for the weak topology of $\mathscr{C}'(G)$ is given by the seminorms $\{\|\cdot\|_f\colon f\in\mathscr{C}(G)\}$, where if T is in $\mathscr{C}'(G)$, $\|T\|_f = |T(f)|$.)

Let $\mathscr{C}'(\hat{G})$ be the dual space of $\mathscr{C}(\hat{G})$. Then

$$\mathscr{C}'(\hat{G}) = \mathscr{C}_0'(\hat{G}) \oplus \mathscr{C}_1'(\hat{G})$$

where $\mathscr{C}_0'(\hat{G})$ and $\mathscr{C}_1'(\hat{G})$ are the dual spaces of $\mathscr{C}_0(\hat{G})$ and $\mathscr{C}_1(\hat{G})$ respectively. Endow $\mathscr{C}'(\hat{G})$, $\mathscr{C}_0'(\hat{G})$, and $\mathscr{C}_1'(\hat{G})$ each with the weak topology induced from $\mathscr{C}(\hat{G})$, $\mathscr{C}_0(\hat{G})$, and $\mathscr{C}_1(\hat{G})$ respectively.

THEOREM 5. *Denote the map $f\to\hat{f}$ of $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$ by $\mathscr{F}f$. Then $\mathscr{F}^*$, the transpose of $\mathscr{F}$, is a topological isomorphism from $\mathscr{C}'(\hat{G})$ onto $\mathscr{C}'(G)$.*

PROOF. The theorem follows directly from the fact that $\mathscr{F}$ is a topological isomorphism. ◻

It is of interest to obtain a slightly different characterization of the space $\mathscr{C}_1'(\hat{G})$. Define the space $\mathscr{S}_1(\hat{G})$ as in §7. For a_1 in $\mathscr{S}_1(\hat{G})$ define the function

$$\check{a}_1 : G \to \mathbf{C}$$

as in §7. Then

$$\check{a}_1(x) = \sum_1 \int_{-\infty}^{\infty} (a_1(\sigma, \Lambda)\Phi_{\tau_2,i_2}, \Phi_{\tau_1,i_1})(\Phi_{\tau_1,i_1}, \pi_{\sigma,\Lambda}(x)\Phi_{\tau_2,i_2}) \cdot \beta(\sigma, \Lambda)\, d\Lambda.$$

$\check{a}_1(x)$ is in $\mathscr{C}(G)$ by Theorem 3′(b).

For a_1 in $\mathscr{S}_1(\hat{G})$ define

$$(Sa_1)(\sigma, \Lambda) = \tfrac{1}{2}[a_1(\sigma, \Lambda) + N_\sigma(\Lambda)^{-1} a_1(\sigma', -\Lambda) N_\sigma(\Lambda)].$$

Then by (7.3)

$$\hat{\check{a}}_1(\sigma, \Lambda) = (Sa_1)(\sigma, \Lambda).$$

Therefore, by Theorem 3′(b) and Lemma 10, the map

$$a_1 \to Sa_1, \qquad a_1 \in \mathscr{S}_1(\hat{G}),$$

is a continuous transformation from $\mathscr{S}_1(\hat{G})$ into $\mathscr{C}_1(\hat{G})$. If a_1 is in $\mathscr{C}_1(\hat{G})$, $Sa_1 = a_1$. Therefore S is a continuous projection from $\mathscr{S}_1(\hat{G})$ onto $\mathscr{C}_1(\hat{G})$. Since $\mathscr{C}_1(\hat{G})$ is the kernel of the continuous map $S - 1$ on $\mathscr{S}_1(\hat{G})$, $\mathscr{C}_1(\hat{G})$ is a closed subspace of $\mathscr{S}_1(\hat{G})$.

Let $\mathscr{S}_1'(\hat{G})$ be the space of distributions on $\mathscr{S}_1(\hat{G})$. We can regard the map S as going from $\mathscr{S}_1(\hat{G})$ to either $\mathscr{S}_1(\hat{G})$ or $\mathscr{C}_1(\hat{G})$. In either case let S^* be its transpose. Then if T is in either $\mathscr{S}_1'(\hat{G})$ or $\mathscr{C}_1'(\hat{G})$, S^*T is a distribution in $\mathscr{S}_1'(\hat{G})$, and

$$(S^*T)(a_1) = T(Sa_1), \qquad a_1 \in \mathscr{S}_1(\hat{G}).$$

THEOREM 6. *S^* is a canonical isomorphism from $\mathscr{C}_1'(\hat{G})$ onto the closed subspace*

$$\overline{\mathscr{S}_1'}(\hat{G}) = \{T \in \mathscr{S}_1'(\hat{G}) : S^*T = T\}$$

of $\mathscr{S}_1'(\hat{G})$.

PROOF. Clearly S^* is a one-to-one map of $\mathscr{C}_1'(G)$ into $\overline{\mathscr{S}_1'}(\hat{G})$. Now suppose that T is in $\overline{\mathscr{S}_1'}(\hat{G})$. We can define a new distribution T_1 in $\mathscr{C}_1'(\hat{G})$ as the restriction of T to the closed subspace $\mathscr{C}_1(\hat{G})$ of $\mathscr{S}_1(\hat{G})$. If a_1 is in $\mathscr{S}_1(\hat{G})$,

$$(S^*T_1)(a_1) = T_1(Sa_1) = T(Sa_1) = (S^*T)(a_1) = Ta_1.$$

Therefore the map

$$S^* : \mathscr{C}_1'(\hat{G}) \to \overline{\mathscr{S}_1'}(\hat{G})$$

is surjective. Our theorem is proved. □

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