Lectures on automorphic L-functions

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PREFACE

This article follows the format of five lectures that we gave on automorphic L-functions. The lectures were intended to be a brief introduction for number theorists to some of the main ideas in the subject. Three of the lectures concerned the general properties of automorphic L-functions, with particular reference to questions of spectral decomposition. We have grouped these together as Part I. While many of the expected properties of automorphic L-functions remain conjectural, a significant number have now been established. The remaining two lectures were focused on the techniques which have been used to establish such properties. These lectures form Part II of the article.

The first lecture (§I.1) is on the standard L-functions for GL_n. Much of this material is familiar and can be used to motivate what follows. In §I.2 we discuss general automorphic L-functions, and various questions that center around the fundamental principle of functoriality. The third lecture (§I.3) is devoted to the spectral decomposition of $L^2(G(F) \setminus G(\mathbb{A}))$. Here we describe a conjectural classification of the spectrum in terms of tempered representations. This amounts to a quantitative explanation for the failure of the general analogue of Ramanujan's conjecture.

There are three principal techniques that we discuss in Part II. The lecture §II.1 is concerned with the trace formula approach and the method of zeta-integrals; it gives only a skeletal treatment of the subject. The lecture §II.2, on the other hand, gives a much more detailed account of the theory of theta-series liftings, including a discussion of counterexamples to the general analogue of Ramanujan's conjecture. We have not tried to relate the counterexamples given by theta-series liftings with the conjectural classification of §I.3. It would be interesting to do so.

These lectures are really too brief to be considered a survey of the subject. There are other introductory articles (references [A.1], [G], [B.1] for Part I) in which the reader can find further information. More detailed discussion is given in various parts of the Corvallis Proceedings and in many of the other references we have cited.

PART I

1 STANDARD L-FUNCTIONS FOR GL_n

Let F be a fixed number field. As usual, F_v denotes the completion of F with respect to a (normalized) valuation v. If v is discrete, $\mathfrak{o}_v$ stands for the ring of integers in F_v, and q_v is the order of the corresponding residue class field. We shall write $\mathbb{A} = \mathbb{A}_F$ for the adèle ring of F.

In this lecture, G will stand for the general linear group GL_n. Then $G(\mathbb{A})$ is the restricted direct product, over all v, of the groups $G(F_v) = GL_n(F_v)$. Thus, $G(\mathbb{A})$ is the topological direct limit of the groups

$$G_S = \prod_{v\in S} G(F_v) \cdot \prod_{v\notin S} G(\mathfrak{o}_v),$$

in which S ranges over all finite sets of valuations of F containing the set S_∞ of Archimedean valuations.

One is interested in the set $\Pi(G(\mathbb{A}))$ of equivalence classes of irreducible, admissible representations of $G(\mathbb{A})$. (Recall that a representation of $G(\mathbb{A})$ is admissible if its restriction to the maximal compact subgroup

$$K = \prod_{v \text{ complex}} U(n,\mathbb{C}) \times \prod_{v \text{ real}} O(n,\mathbb{R}) \times \prod_{v \text{ discrete}} GL_n(\mathfrak{o}_v)$$

contains each irreducible representation of K with only finite multiplicity.) Similarly, one has the set $\Pi(G(F_v))$ of equivalence classes of irreducible admissible representations of $G(F_v)$. It is known [F] that any $\pi \in \Pi(G(\mathbb{A}))$ can be decomposed into a restricted tensor product

$$\bigotimes_v \pi_v, \qquad \pi_v \in \Pi(G(F_v)),$$

of irreducible, admissible representations of the local groups.

The *unramified principal series* is a particularly simple subset of $\Pi(G(F_v))$ to describe. Suppose that the valuation v is discrete. One has the Borel subgroup

$$B(F_v) = \{b = \begin{pmatrix} b_1 & \cdots & * \\ & \ddots & \vdots \\ O & & b_n \end{pmatrix}\} \subseteq G(F_v)$$

of $G(F_v)$, and for any n-tuple $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$b \longrightarrow \chi_z(b) = |b_1|_v^{z_1} \cdots |b_n|_v^{z_n}$$

gives a quasi-character on $B(F_v)$. Let $\tilde{\pi}_{v,z}$ be the representation of $G(F_v)$ obtained by inducing χ_z from $B(F_v)$ to $G(F_v)$. (Recall that $\tilde{\pi}_{v,z}$ acts on the space of locally constant functions ϕ on $G(F_v)$ such that

$$\phi(bz) = \chi_z(b)(\prod_{i=0}^{n-1} |b_i|_v^{\frac{n-1}{2}-i})\phi(x), \qquad b \in B(F_v),\ x \in G(F_v),$$

and that

$$(\tilde{\pi}_{v,z}(y)\phi)(x) = \phi(xy)$$

for any such ϕ.) We shall assume that

$$Re(z_1) \geq Re(z_2) \geq \cdots \geq Re(z_n).$$

It is then a very special case of the Langlands classification [B-W, XI, §2] that $\tilde{\pi}_{v,z}$ has a unique irreducible quotient $\pi_{v,z}$. The representations $\{\pi_{v,z}\}$ obtained in this way are the unramified principal series. They are precisely the representations in $\Pi(G(F_v))$ whose restrictions to $G(\mathfrak{o}_v)$ contain the trivial representation. If π_v is any representation in $\Pi(G(F_v))$ which is equivalent to some $\pi_{v,z}$, it makes sense to define a semisimple conjugacy class

$$\sigma(\pi_v) = \begin{pmatrix} q_v^{-z_1} & & 0 \\ & \ddots & \\ 0 & & q_v^{-z_n} \end{pmatrix}$$

in $GL_n(\mathbb{C})$. For $\sigma(\pi_v)$ does depend only on the equivalence class of π_v; conversely, if two such representations are inequivalent, the corresponding conjugacy classes are easily seen to be distinct.

Suppose that $\pi = \bigotimes_v \pi_v$ is a representation in $\Pi(G(\mathbb{A}))$. Since π is admissible, almost all the local constituents π_v belong to the unramified principal series. Thus, π gives rise to a family

$$\sigma(\pi) = \{\sigma_v(\pi) = \sigma(\pi_v) :\ v \notin S\}$$

of semisimple conjugacy classes in $GL_n(\mathbb{C})$, which are parametrized by the valuations outside of some finite set $S \supseteq S_\infty$. Bearing in mind that a semisimple conjugacy class in $GL_n(\mathbb{C})$ is determined by its characteristic polynomial, one defines the local L-functions

$$L_v(s, \pi) = L(s, \pi_v) = \det(1 - \sigma_v(\pi)q_v^{-s})^{-1}, \qquad s \in \mathbb{C},\ v \notin S.$$

The global L-function is then given as a formal product

$$L_S(s,\pi) = \prod_{v \notin S} L_v(s,\pi).$$

If the global L-function is to have interesting arithmetic properties, one needs to assume that π is automorphic. We shall first review the notion of an automorphic representation, and then describe the properties of the corresponding automorphic L-functions.

The group $G(F)$ embeds diagonally as a discrete subgroup of

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : \ |\det\ g| = 1\}.$$

The space of *cusp forms* on $G(\mathbb{A})^1$ consists of the functions $\phi \in L^2(G(F) \setminus G(\mathbb{A})^1)$ such that

$$\int_{N_P(F)\setminus N_P(\mathbb{A})} \phi(nx)dn \ = \ 0$$

for almost all $x \in G(\mathbb{A})^1$, and for the unipotent radical N_P of any proper, standard parabolic subgroup P. (Recall that standard parabolic subgroups are subgroups of the form

$$P(\mathbb{A}) = \left\{p = \begin{pmatrix} p_1 & \cdots & * \\ & \ddots & \vdots \\ O & & p_r \end{pmatrix} : p_k \in GL_{n_k}\right\},$$

where $(n_1,\dots,n_r)$ is a partition of n.) The space of cusp forms is a closed, right $G(\mathbb{A})^1$-invariant subspace of $L^2(G(F) \setminus G(\mathbb{A})^1)$, which is known to decompose into a discrete direct sum of irreducible representations of $G(\mathbb{A})^1$. A representation $\pi \in \Pi(G(\mathbb{A}))$ is said to be *cuspidal* if its restriction to $G(\mathbb{A})^1$ is equivalent to an irreducible constituent of the space of cusp forms. We note that such a representation need not be unitary; indeed, if π is cuspidal, so are all the representations $\{\pi \otimes |\det|^z : \ z \in \mathbb{C}\}$. Now, suppose that $p \in P(\mathbb{A})$ is as above, with P a given standard parabolic subgroup, and that for each i, $1 \leq i \leq r$, π_i is a cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$. Then

$$p \longrightarrow \pi_1(p_1) \otimes \cdots \otimes \pi_r(p_r)$$

is a representation of $P(\mathbb{A})$, which we can induce to $G(\mathbb{A})$. The automorphic representations of $G(\mathbb{A})$ are the irreducible constituents of induced representations of this form [L.4]. We shall denote the subset of automorphic representations in $\Pi(G(\mathbb{A}))$ by $\Pi_{\text{aut}}(G)$.

Suppose that $\pi \in \Pi_{\text{aut}}(G)$. It is easily seen that the infinite product for $L_S(s,\pi)$ converges in some right half plane. Moreover, Godement and Jacquet

[G-J] have shown that $L_S(s,\pi)$ has analytic continuation to a meromorphic function of $s \in \mathbb{C}$ which satisfies a functional equation. Their method is a generalization from GL_1 of the method of Tate's thesis [T.1], and will be described in §II.1.

It is useful to consider certain subsets of $\Pi_{\mathrm{aut}}(G)$. Let $\Pi_{\mathrm{disc}}(G)$ denote the set of irreducible, *unitary* representations of $G(\mathbb{A})$ whose restriction to $G(\mathbb{A})^1$ occurs discretely in $L^2(G(F) \setminus G(\mathbb{A})^1)$. This contains the set $\Pi_{\mathrm{cusp}}(G)$ of irreducible, *unitary* cuspidal representations of $G(\mathbb{A})$. We shall then write $\Pi(G)$ simply for the set of irreducible representations of $G(\mathbb{A})$ obtained by inducing representations

$$p \longrightarrow \pi_1(p_1) \otimes \cdots \otimes \pi_r(p_r), \qquad \pi_i \in \Pi_{\mathrm{disc}}(GL_{n_i}),$$

from standard parabolic subgroups. (It is a peculiarity of GL_n that such unitary induced representations are already irreducible. In general, one must define $\Pi(G)$ to be the set of irreducible constituents of these induced representations.) The representations $\Pi(G)$ are precisely the ones which occur in the spectral decomposition of $L^2(G(F) \setminus G(\mathbb{A}))$. This deep fact is a consequence of the theory of Eisenstein series, initiated by Selberg [S], and established for general groups by Langlands [L.3]. A second major consequence of the theory of Eisenstein series is that the representations in $\Pi_{\mathrm{disc}}(G)$ and $\Pi(G)$ are automorphic. Taking this fact for granted, we obtain an embedded sequence

$$\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G) \subset \Pi(G) \subset \Pi_{\mathrm{aut}}(G) \subset \Pi(G(\mathbb{A}))$$

of families of irreducible representations of $G(\mathbb{A})$.

The representations in $\Pi(G)$ have a striking rigidity property.

Theorem The map

$$\pi \longrightarrow \sigma(\pi) = \{\sigma_v(\pi) : v \notin S\}, \qquad \pi \in \Pi(G),$$

from $\Pi(G)$ to families of semisimple conjugacy classes in $GL_n(\mathbb{C})$, is injective. In other words, a representation in $\Pi(G)$ is completely determined by the associated family of conjugacy classes.

For cuspidal representations, this theorem is closely related to the original multiplicity one theorem [Sh]. The extension to $\Pi(G)$ follows from analytic properties of the corresponding L-functions [J-S, Theorem 4.4] and the recent classification [M-W] by Moeglin and Waldspurger of the discrete spectrum of GL_n.

The theorem is reminiscent of a similar rigidity property of representations of Galois groups. Suppose that

$$r : Gal(\overline{F}/F) \longrightarrow GL_n(\mathbb{C})$$

is a continuous representation of the Galois group of the algebraic closure $\overline{F}$ of F. For every valuation v outside a finite set $S \supseteq S_\infty$, there is an associated Frobenius conjugacy class in the image of the Galois group. This gives a semisimple conjugacy class $\sigma_v(r)$ in $GL_n(\mathbb{C})$. It is an immediate consequence of the Tchebotarev density theorem that r is completely determined by the family

$$\sigma(r) = \{\sigma_v(r) : v \notin S\}.$$

We should also recall the local and global Artin L-functions

$$L_v(s,r) = \det(1 - \sigma_v(r)q_v^{-s})^{-1}, \qquad s \in \mathbb{C},\ v \notin S,$$

and

$$L_S(s,r) = \prod_{v \notin S} L_v(s,r),$$

attached to r.

Some years ago, Langlands conjectured [L.1] that the similarity between the two types of L-function was more than just formal.

Conjecture (Langlands) For any continuous representation

$$r : Gal(\overline{F}/F) \longrightarrow GL_n(\mathbb{C})$$

of the Galois group there is an automorphic representation $\pi \in \Pi(G)$, necessarily unique, such that $\sigma_v(\pi) = \sigma_v(r)$ for all v outside some finite set $S \supseteq S_\infty$. In particular

$$L_S(s,\pi) = L_S(s,r).$$

The conjecture represents a fundamental problem in number theory. The case $n = 1$ is just the Artin reciprocity law, which is of course known, but highly nontrivial. There has also been significant progress in the case $n = 2$. If the image of r in $GL_2(\mathbb{C})$ is a dihedral group, the conjecture follows from the converse theorem of Hecke theory [J-L] or from the properties of the Weil representation [S-T]. (See §II.1.) If r is an irreducible 2-dimensional representation which is not dihedral, its image in $PGL_2(\mathbb{C}) \cong SO(3,\mathbb{C})$ will be either tetrahedral, octahedral or icosohedral. These cases are much deeper, but the first two have been solved [L.6], [Tu]. The essential new ingredient was

Langlands' solution of the (cyclic) base change problem for GL_2. For general n, the base change problem was solved recently by Arthur and Clozel [A-C]. This leads to an affirmative answer to the conjecture for any representation r whose image is nilpotent.

We shall describe the base change theorem in more detail. Suppose that E/F is a Galois extension, with cyclic Galois group

$$Gal(E/F) = \{1, \gamma, \gamma^2, \ldots, \gamma^{\ell-1}\}$$

of prime order ℓ. Then there is a short exact sequence

$$1 \longrightarrow Gal(\overline{E}/E) \longrightarrow Gal(\overline{F}/F) \longrightarrow Gal(E/F) \longrightarrow 1$$

of Galois groups. If r is a representation of $Gal(\overline{F}/F)$, let r_E be the restriction of r to the subgroup $Gal(\overline{E}/E)$. Suppose that $v \notin S$ is a valuation which is unramified for both r and E/F, and that $v_E = v \circ \mathrm{Norm}_{E/F}$ is the associated function on E. It is easy to check that the conjugacy classes in $GL_n(\mathbb{C})$ are related by

$$\sigma_{V_i}(r_E) = \sigma_v(r)$$

if $v_E = V_1 \cdots V_\ell$ splits completely in E, and

$$\sigma_V(r_E) = \sigma_v(r)^\ell$$

if $v_E = V$ remains prime in E. Another way to say this is that

$$L_{S_E}(s, r_E) = \prod_{j=1}^{\ell} L_S(s, r \otimes \varepsilon^j),$$

where S_E is the set of valuations of E over S, and ε is the one dimensional representation

$$\gamma^k \longrightarrow e^{\frac{2\pi i k}{\ell}}, \qquad k = 1, \ldots, \ell,$$

of $Gal(E/F)$. Thus, the map $r \longrightarrow r_E$ from n-dimensional representations of $Gal(\overline{F}/F)$ to n-dimensional representations of $Gal(\overline{E}/E)$ is determined in a simple way by its behaviour on Frobenius conjugacy classes. Moreover, it is easy to check that an arbitrary n-dimensional representation R of $Gal(\overline{E}/E)$ is of the form r_E if and only if the conjugate R^γ of R by γ is equivalent to R.

Langlands' conjecture suggests that there should be a parallel operation on automorphic representations. Let $G_E = GL_{n,E}$ denote the general linear group, regarded as an algebraic group over E.

Theorem [A-C] There is a canonical map $\pi \to \pi_E$ from $\Pi(G)$ to $\Pi(G_E)$ such that

$$\sigma_{V_i}(\pi_E) = \sigma_v(\pi)$$

if $v_E = V_1 \cdots V_\ell$ splits completely in E and

$$\sigma_V(\pi_E) = \sigma_v(\pi)^\ell$$

if $v_E = V$ remains prime in E. In particular

$$L_{S_E}(s, \pi_E) = \prod_{j=1}^{\ell} L_S(s, \pi \otimes (\eta \circ \det)^j),$$

where η is the Grössencharacter of F associated to ε by class field theory. Moreover, an automorphic representation $\Pi \in \Pi(G_E)$ is of the form π_E if and only if $\Pi^\gamma \cong \Pi$.

We have already mentioned that the theorem was proved for $n = 2$ by Langlands [L.6], who built on earlier work of Saito and Shintani. The proof for general n in [A-C] actually applies only to a subset of $\Pi(G)$, namely representations 'induced from cuspidal'. However, the general case follows easily from this and the classification [M-W] of the discrete spectrum of GL_n. The proof of base change relies in an essential way on the trace formula for GL_n.

We have so far treated the simplest case of unramified primes. We should say a few words about the ramified places before we go on to more general L-functions. In [G-J], Godement and Jacquet define a local L-function $L(s, \pi_v)$ and ε-factor $\varepsilon(s, \pi_v, \psi_v)$ for any admissible representation $\pi_v \in \Pi(G(F_v))$ and any nontrivial additive character ψ_v of F_v. They then define the global L-function

$$L(s, \pi) = \prod_v L(s, \pi_v)$$

and ε-factor

$$\varepsilon(s, \pi) = \prod_v L(s, \pi_v)$$

as products over all places v. Here $\pi \in \Pi_{\mathrm{aut}}(G)$ is any automorphic representation, and $\psi = \bigotimes_v \psi_v$ is a nontrivial additive character on $\mathbb{A}/F$. Since the local root numbers are trivial at unramified places, the global root number is defined as a finite product. It is independent of ψ. The main result of [G-J] is

Theorem [G-J] Suppose that $\pi \in \Pi_{\mathrm{cusp}}(G)$ is a cuspidal representation with contragradient $\tilde{\pi}$. Then $L(s, \pi)$ can be analytically continued as a meromorphic function of $s \in \mathbb{C}$ which satisfies the functional equation

$$L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi}).$$

The function $L(s, \pi)$ is entire unless $n = 1$ and π is an unramified character. □

2 GENERAL AUTOMORPHIC L-FUNCTIONS.

From now on, G will be an arbitrary reductive algebraic matrix group defined over F. The objects $G(\mathbb{A})$, $G(\mathbb{A})^1$, $\Pi(G(F_v))$, $\Pi(G(\mathbb{A}))$, etc., are defined essentially as above. Using the standard parabolic subgroups of G as we did for GL_n, we can also define the families

$$\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G) \subset \Pi(G) \subset \Pi_{\mathrm{aut}}(G) \subset \Pi(G(\mathbb{A}))$$

of irreducible representations of $G(\mathbb{A})$. By the general theory of Eisenstein series [L.3], $\Pi(G)$ is precisely the set of irreducible representations which occur in the spectral decomposition of $L^2(G(F) \setminus G(\mathbb{A}))$.

If $\pi = \bigotimes_v \pi_v$ is any representation in $\Pi(G(\mathbb{A}))$, it can be shown that π_v is unramified for all v outside a finite set $S \supset S_\infty$. (This means that G is quasi-split over F_v and split over an unramified extension, and that π_v has a fixed vector under a hyperspecial maximal compact subgroup of $G(F_v)$. As with GL_n, any unramified representation is a constituent of the representation induced from an unramified quasi-character, essentially uniquely determined, on a Borel subgroup defined over F_v. See [B, §10.4].) What plays the role for general G of the conjugacy classes $\sigma(\pi_v)$ in $GL_n(\mathbb{C})$?

To take the place of $GL_n(\mathbb{C})$, Langlands [L.1] introduced a certain complex, nonconnected group. In its simplest form, this *L-group* is a semi-direct product

$${}^L G = \hat{G} \rtimes Gal(E/F),$$

where $\hat{G}$ is a complex reductive group which is 'dual' to G, and E/F is any finite Galois extension over which G splits. The action of $Gal(E/F)$ on $\hat{G}$ is determined in a canonical way, up to inner automorphism, from the action of the Galois group on the Dynkin diagram of G. Rather than define the L-group precisely, we shall simply note that it comes with some extra structure, which in essence determines it uniquely. Suppose that $T \subset B \subset G$ and $\hat{T} \subset \hat{B} \subset \hat{G}$ are maximal tori, embedded in Borel subgroups of G and $\hat{G}$. The L-group is then equipped with an isomorphism from $\hat{T}$ onto the complex dual torus $X^*(T) \otimes \mathbb{C}^*$ of T, which maps the simple roots $\hat{\Delta}$ of $(\hat{B}, \hat{T})$ onto the simple co-roots $\Delta^\vee$ of (G, T), and which is compatible with the canonical actions of $Gal(E/F)$. (See [K.1, §1].) The simplest examples of pairs $(G, \hat{G})$ are $(GL_n, GL_n(\mathbb{C}))$, $(SL_n, PGL_n(\mathbb{C}))$, $(PGL_n, SL_n(\mathbb{C}))$, $(SO_{2n+1}, Sp_{2n}(\mathbb{C}))$, $(Sp_{2n}, SO_{2n+1}(\mathbb{C}))$, $(SO_{2n}, SO_{2n}(\mathbb{C}))$. In each of these cases, G is already split, and the field E may be taken to be F.

The unramified representations have the following striking characterization in terms of the L-group [B, §10.4]. For almost all places v, G is quasi-split over F_v and split over an unramified extension, and $G(\mathfrak{o}_v)$ is a hyperspecial maximal compact subgroup. For any such v, the representations $\pi_v \in \Pi(G(F_v))$ which have a $G(\mathfrak{o}_v)$-fixed vector are in one-to-one correspondence with the semisimple conjugacy classes $\sigma(\pi_v)$ in ${}^LG$ whose projection onto the factor $Gal(E/F)$ equals the Frobenius class at v. Thus, a representation $\pi = \bigotimes_v \pi_v$ in $\Pi(G(\mathbb{A}))$ gives rise to a family

$$\sigma(\pi) = \{\sigma_v(\pi) = \sigma(\pi_v) : v \notin S\}$$

of semisimple conjugacy classes in ${}^LG$.

In order to define an automorphic L-function, one needs to take a finite dimensional representation

$$r : {}^LG \longrightarrow GL_n(\mathbb{C})$$

of the L-group as well as an automorphic representation $\pi \in \Pi_{\text{aut}}(G)$. This gives rise to a family

$$\{r(\sigma_v(\pi)) : \ v \notin S\}$$

of semisimple conjugacy classes in $GL_n(\mathbb{C})$. The general automorphic L-function is then defined as the product

$$L_S(s, \pi, r) = \prod_{v \notin S} \det(1 - r(\sigma_v(\pi) q_v^{-s})^{-1}, \quad s \in \mathbb{C}.$$

It is not hard to verify that the product converges in some right half plane. Again, one expects the L-functions to have analytic continuation and functional equation, although this is still far from known in general. Bear in mind that we are free to let the extension E/F be arbitrarily large. Therefore, the general automorphic L-functions include both the Artin L-functions and the standard L-functions for GL_n discussed in §1.

Examples

1. Suppose that E/F is a cyclic extension of prime order ℓ. Take G to be $Res_{E/F}(GL_{n,E})$, the group obtained from the general linear group over E by restriction of scalars. This is perhaps the simplest example after GL_n itself. The L-group is given by

$$^{L}G = \underbrace{(GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C}))}_{\ell} \rtimes Gal(E/F),$$

where the cyclic Galois group acts by permuting the factors. There is a canonical representation

$$r : {}^{L}G \longrightarrow GL_{n\ell}(\mathbb{C}),$$

in which $\hat{G}$ is embedded diagonally, and $Gal(E/F)$ is mapped into the obvious group of permutation matrices. Since $G(\mathbb{A}) \cong GL_n(\mathbb{A}_E)$, an automorphic representation $\pi \in \Pi_{\text{aut}}(G)$ can be identified with an automorphic representation $\Pi \in \Pi_{\text{aut}}(GL_{n,E})$ of the general linear group over E. One can check that

$$L_S(s, \pi, r) = L_{S_E}(s, \Pi).$$

2. Suppose that $G = GL_n \times GL_m$. There is a canonical representation

$$r : GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \longrightarrow GL_{nm}(\mathbb{C}).$$

An automorphic representation $\pi \in \Pi_{\text{aut}}(G)$ is a tensor product of automorphic representations $\pi_1 \in \Pi_{\text{aut}}(GL_n)$ and $\pi_2 \in \Pi_{\text{aut}}(GL_m)$. The corresponding L-function $L_S(s, \pi, r)$ equals the general Rankin–Selberg product $L_S(s, \pi_1 \times \tilde{\pi}_2)$. Its analytic continuation and functional equation have been established by Jacquet, Piatetskii-Shapiro, and Shalika [J-P-S], and will be discussed in §II.1.

3. Suppose that G is one of the classical groups SO_{2n+1}, Sp_{2n} or SO_{2n}. Then $\hat{G}$ equals $Sp_{2n}(\mathbb{C}), SO_{2n+1}(\mathbb{C})$, or $SO_{2n}(\mathbb{C})$ respectively. In each case, there is a standard embedding r of $\hat{G}$ into a complex general linear group. The corresponding L-functions $L_S(s, \pi, r)$ have been studied by Piatetskii-Shapiro and Rallis [P-R], and will also be discussed in §II.1.

Underlying everything is the fundamental problem of establishing Langlands' functoriality principle [L.1], [B]. This pertains to maps $\rho : {}^{L}G \longrightarrow {}^{L}G'$ between two L-groups. We shall say that such a map is an *L-homomorphism* if E' is a subfield of E, and if the composition of ρ with the projection of ${}^{L}G'$ onto $Gal(E'/F)$ equals the canonical map of $Gal(E/F)$ onto $Gal(E'/F)$.

Conjecture (Langlands) Suppose that G and G' are reductive groups over F, that G' is quasi-split, and that $\rho : {}^LG \longrightarrow {}^LG'$ is an L-homomorphism between their L-groups. Then for any automorphic representation $\pi \in \Pi_{\mathrm{aut}}(G)$, there is an automorphic representation $\pi' \in \Pi_{\mathrm{aut}}(G')$ such that $\rho(\sigma_v(\pi)) = \sigma_v(\pi')$ for all v outside a finite set $S \supset S_\infty$. In particular,

$$L_S(s, \pi, r \circ \rho) \;=\; L_S(s, \pi', r)$$

for any finite dimensional representation r of ${}^LG'$.

Remarks

1. Suppose that π belongs to the subset $\Pi(G)$ of $\Pi_{\mathrm{aut}}(G')$. Then one should be able to choose π' in $\Pi(G')$. This question is related to the discussion in §3.
2. Suppose that $G = \{1\}$ and $G' = GL_n$. Then an L-homomorphism between their L-groups is an n-dimensional representation of $Gal(E/F)$. The functoriality principle becomes the conjecture stated in §1, relating Artin L-functions to the automorphic L-functions of GL_n. We have already discussed the limited number of cases in which it has been solved. This apparently simple case illustrates the depth of the general functoriality principle.

Examples

1. (a) (Base change). Let E/F be a cyclic extension of prime degree ℓ. Set $G = GL_n$ and $G' = Res_{E/F}(GL_{n,E})$. Define an L-homomorphism $\rho : {}^LG \longrightarrow {}^LG'$ by taking the diagonal embedding

   $$\hat{G} \;=\; GL_n(\mathbb{C}) \;\hookrightarrow\; \underbrace{(GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C}))}_{\ell} \;=\; \hat{G}'.$$

   Functoriality in this case asks for a correspondence from $\Pi_{\mathrm{aut}}(G)$ to

   $$\Pi_{\mathrm{aut}}(G') = \Pi_{\mathrm{aut}}(GL_{n,E}).$$

   This follows easily from the base change theorem stated in §1.

   (b) (Automorphic induction). Let G' be as above and take G'' to be $GL_{n\ell}$. Let

   $$\rho' : {}^LG' \;\longrightarrow\; GL_{n\ell}(\mathbb{C}) \;=\; {}^LG''$$

   be the representation defined in the previous set of examples. This case of functoriality is also known. It was proved (in a slightly different form) in [A-C], as a consequence of base change.

2. Set $G = GL_n \times GL_m$, $G' = GL_{nm}$ and

$$\rho : {}^L G \cong GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \longrightarrow GL_{nm}(\mathbb{C}) \cong {}^L G'.$$

It would be extremely valuable to have functoriality for this example. However, it is very deep, and is far from being proved.

3. Suppose that ${}^L G$ is one of the complex classical groups $Sp_{2n}(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$ or $SO_{2n}(\mathbb{C})$, and that ρ is the standard representation. Functoriality is not known in this case, but one hopes that it will eventually follow from the twisted trace formula for the general linear group, and the developing theory of endoscopy. There is also another approach, using L-functions, which will be discussed briefly in §II.1.

We have continued to emphasize mainly the unramified places and the associated conjugacy classes. This is the simplest way to describe things, at least initially. However, it eventually becomes necessary to deal with the ramified places as well.

To this end, we first recall that the local Weil group W_{F_v} is defined for any valuation v. It fits into the commutative diagram

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & Gal(\overline{F}_v/F_v) \\ \downarrow & & \downarrow \\ W_F & \longrightarrow & Gal(\overline{F}/F) \end{array}$$

of locally compact groups, in which W_F denotes the global Weil group. The vertical embeddings are determined only up to conjugacy. One can also define the (local) Weil–Deligne group [K.1, §12] in the form

$$L_{F_v} = \begin{cases} W_{F_v}, & \text{if } v \text{ is Archimedean,} \\ W_{F_v} \times SU(2,\mathbb{R}), & \text{if } v \text{ is discrete.} \end{cases}$$

Langlands and Deligne have defined local L-functions $L(s, r_v)$ and root numbers $\varepsilon(s, r_v, \psi_v)$ for any finite dimensional representation r_v of either W_{F_v} or L_{F_v}, and for any nontrivial additive character ψ_v on F_v. (See [T.2].)

It is actually best to define the L-group of G as

$${}^L G = \hat{G} \rtimes W_F,$$

where W_F acts on $\hat{G}$ through its projection onto $Gal(E/F)$. One can also define local L-groups

$$^LG_v = \hat{G} \rtimes W_{F_v}.$$

These come equipped with embeddings

$$^LG_v \hookrightarrow {}^LG,$$

which are determined up to conjugacy in ^LG. The notion of L-homomorphism can be extended in the obvious way to maps between local or global L-groups, or indeed to any maps between locally compact groups which both fibre over W_{F_v} or W_F.

The local Langlands conjecture can be stated informally as follows.

Conjecture (Langlands [L.1], [B]) There is a partition of the admissible representations $\Pi(G(F_v))$ into finite disjoint subsets Π_{ϕ_v} which are parametrized by the $\hat{G}$-orbits of admissible L-homomorphisms

$$\phi_v : L_{F_v} \longrightarrow {}^LG_v.$$

The definition of an *admissible* L-homomorphism ϕ_v, which we have omitted, is straightforward. One simply imposes several natural conditions, the most significant one being that if the image of ϕ_v is contained in a parabolic subgroup of ^LG, then the corresponding parabolic subgroup of G must be defined over F [B, §8.2]. The conjectured partition should have a number of natural properties [B, §10]. For example, if ϕ_v is unramified in the obvious sense, and corresponds to a semisimple conjugacy class Φ in ^LG_v, then Π_{ϕ_v} should consist of the set of unramified representations $\pi_v \in \Pi(G(F_v))$ such that $\sigma(\pi_v) = \Phi$. However, the expected properties as they are presently conceived are not strong enough to determine the partition uniquely. The local Langlands conjecture has been established in the following cases.

(i) G is a torus [L.2].
(ii) F is Archimedean [L.7].
(iii) $G = GL_n$, and n is prime to q_v [My].
(iv) $G = GL_p$, with p prime [K-M].

Assume for the moment that the local conjecture has been established. Then any representation $\pi_v \in \Pi(G(F_v))$ belongs to a unique packet Π_{ϕ_v}. In this context, the functoriality principle can be described as follows. Suppose that G' is quasi-split, and that

$$\rho : {}^LG \longrightarrow {}^LG'$$

is an L-homomorphism. For each v this determines a commutative diagram

$$\begin{array}{ccc} {}^L G_v & \xrightarrow{\rho_v} & {}^L G'_v \\ \downarrow & & \downarrow \\ {}^L G & \xrightarrow{\rho} & {}^L G'. \end{array}$$

Suppose that $\pi = \bigotimes_v \pi_v$ is a representation in $\Pi_{\mathrm{aut}}(G)$. We are assuming that each π_v belongs to a uniquely determined packet Π_{ϕ_v}. For each v, $\rho_v \circ \phi_v$ is then an admissible homomorphism from L_{F_v} to ${}^L G'_v$. The functoriality principle is that there is an automorphic representation $\pi' = \bigotimes_v \pi'_v$ in $\Pi_{\mathrm{aut}}(G')$ such that for each v, π'_v belongs to the packet $\pi_{\rho_v \circ \phi_v}$.

Suppose that

$$r : {}^L G \longrightarrow GL_n(\mathbb{C})$$

is a finite dimensional representation of the global L-group, and that

$$\pi \;=\; \bigotimes_v \pi_v, \qquad\qquad \pi_v \in \Pi_{\phi_v},$$

belongs to $\Pi_{\mathrm{aut}}(G)$. If r_v is the composition of the embedding ${}^L G_v \hookrightarrow {}^L G$ with r, $r_v \circ \phi_v$ is an n-dimensional representation of L_{F_v}. One can thus define the local L-functions

$$L(s, \pi_v, r_v) \;=\; L(s, r_v \circ \phi_v)$$

and ε-factors

$$\varepsilon(s, \pi_v, r_v, \psi_v) \;=\; \varepsilon(s, r_v \circ \phi_v, \psi_v)$$

for all places v. The global L-functions

$$L(s, \pi, r) \;=\; \prod_v L(s, \pi_v, \rho_v)$$

and ε-factors

$$\varepsilon(s, \pi, r) \;=\; \prod_v \varepsilon(s, \pi_v, r_v, \psi_v)$$

can then be defined as products over all v. As before, $\psi = \bigotimes_v \psi_v$ is a nontrivial additive character on $\mathbb{A}/F$. The global automorphic L-functions are expected to have analytic continuation, and to satisfy the functional equation

$$L(s, \pi, r) = \varepsilon(s, \pi, r) L(1 - s, \pi, \tilde{r}).$$

When he first defined these objects [L.1], Langlands pointed out that the analytic continuation and functional equation would follow from the general functoriality principle and the theorem for GL_n that was later proved by Godement and Jacquet. One would just apply functoriality with $G' = GL_n$, and with

$$\rho = r \times 1 \;:\; {}^L G \;=\; \hat{G} \rtimes W_F \;\longrightarrow\; GL_n(\mathbb{C}) \times W_F.$$

3 UNIPOTENT AUTOMORPHIC REPRESENTATIONS

We shall conclude with a conjectural explanation for the failure of the analogue of Ramanujan's conjecture for general G. This amounts to a classification of the representations in $\Pi(G)$ in terms of tempered representations.

The problem can be motivated from a different point of view. For general G, the strong multiplicity one theorem fails. In other words, the map

$$\sigma : \pi \longrightarrow \sigma(\pi) = \{\sigma_v(\pi) : v \notin S\},$$

from $\Pi(G)$ to families of semisimple conjugacy classes in ${}^LG$, is not injective. (Let us agree to identify two elements $\sigma(\pi)$ and $\sigma(\pi')$ in the image if $\sigma_v(\pi) = \sigma_v(\pi')$ for almost all v.) One could look for some equivalence relation on $\Pi(G)$, defined without reference to σ, whose classes are contained in the fibres of σ. The original idea for such an equivalence relation is due to Langlands, and is now part of the theory of endoscopy.

Suppose that

$$\phi : W_F \longrightarrow {}^LG$$

is an L-homomorphism from the global Weil group into ${}^LG$ which is admissible; that is, each of the corresponding local maps

$$\phi_v : L_{F_v} \twoheadrightarrow W_{F_v} \longrightarrow {}^LG_v$$

is admissible. Then according to the local Langlands conjecture, there are finite packets Π_{ϕ_v} in $\Pi(G(F_v))$, and from these one can form a global packet

$$\Pi_\phi = \{\pi = \bigotimes_v \pi_v \in \Pi(G(\mathbb{A})) : \pi_v \in \Pi_{\phi_v}\}.$$

It would be a consequence of the general functoriality principle that Π_ϕ actually contains a representation in $\Pi_{\text{aut}}(G)$. However, one would ultimately like to have more precise information. Suppose that the image of ϕ in $\hat{G}$ is *bounded*. (If $G = GL_n$, this means that ϕ corresponds to a *unitary* representation of W_F.) Then there is a conjectural formula, which is implicit in the paper [L-L] of Labesse and Langlands, for the multiplicity with which any representation $\pi \in \Pi_\phi$ occurs in $L^2(G(F) \setminus G(\mathbb{A}))$. Observe that the admissibility of the representations $\pi \in \Pi_\phi$ is built into the definition. This implies that for almost all v, π_v is the unique representation in Π_{ϕ_v} which has a $G(\mathfrak{o}_v)$-fixed vector. It follows that the map σ is constant on Π_ϕ.

We shall not recall the definition of a tempered representation, beyond noting that a global packet Π_ϕ should consist of tempered representations precisely when the image of ϕ in $\hat{G}$ is bounded. This is one of the required properties of the conjectural local correspondence. The classical Ramanujan conjecture can be regarded as an assertion that certain representations of GL_2 are tempered. Now the multiplicity formulas in [L-L] were intended only for tempered representations. But it is known that for general G there are many representations in $\Pi_{\text{cusp}}(G)$ which are not tempered. Such examples were first discovered for Sp_4, by Kurokawa [Ku] and Howe and Piatetskii-Shapiro [H-P]. How can one account for these objects?

We shall begin by describing the theorem of Moeglin and Waldspurger, which gives a classification of the discrete spectrum of GL_n in terms of cuspidal representations.

Theorem [M-W] There is a bijection between the set of representations $\pi \in \Pi_{\text{disc}}(GL_n)$ and the set of pairs (d, τ), in which d is a divisor of n and τ is a representation in $\Pi_{\text{cusp}}(GL_d)$.

For a given pair (d, τ), the corresponding $\pi \in \Pi_{\text{disc}}(GL_n)$ is constructed as follows. Set

$$P(\mathbb{A}) = \{p = \begin{pmatrix} p_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & p_m \end{pmatrix} : \quad p_i \in GL_d(\mathbb{A})\}$$

where $n = dm$, and let $\tilde{\pi}$ be the representation obtained by inducing the representation

$$p \rightarrow \tau(p_1)|\det p_1|^{\frac{m-1}{2}} \otimes \tau(p_2)|\det p_2|^{\frac{m-3}{2}} \otimes \cdots \otimes \tau(p_m)|\det p_m|^{-\frac{(m-1)}{2}}$$

from $P(\mathbb{A})$ to $G(\mathbb{A})$. Then π is the unique irreducible quotient of $\tilde{\pi}$. In particular,

$$L(s, \pi) = L(s + \tfrac{m-1}{2}, \tau)L(s + \tfrac{m-3}{2}, \tau) \cdots L(s - \tfrac{m-1}{2}, \tau).$$

Notice that this formula provides an extension of the analyticity assertion of Godement–Jacquet to representations in the discrete spectrum of GL_n; $L(s, \pi)$ will be entire unless $d = 1$ and τ is an unramified character. Now, it is not hard to see that π is nontempered if $m > 1$. Conversely if $m = 1$, so that π belongs to $\Pi_{\text{cusp}}(GL_n)$, then π is expected to be tempered. This is the generalized Ramanujan conjecture, which is believed to hold for GL_n. The theorem can therefore be interpreted as a description of $\Pi_{\text{disc}}(GL_n)$ in terms of tempered representations. It is this interpretation which should carry over to other groups.

The classification of $\Pi_{\mathrm{disc}}(GL_n)$ has a description in terms of the global parameters $\phi : W_F \to {}^LG$. In the case of GL_n, a packet Π_ϕ should contain only one representation, and this in turn should belong to $\Pi_{\mathrm{cusp}}(GL_n)$ precisely when the parameter ϕ, regarded as an n-dimensional representation of W_F, is irreducible and unitary. However, the map $\phi \to \pi$ is not surjective. There are many representations $\pi \in \Pi_{\mathrm{cusp}}(GL_n)$ which do not correspond to any n-dimensional representation of the global Weil group. With this difficulty in mind, Langlands [L.5, §2] suggested that the tempered representations in the sets

$$\Pi(GL_n), \qquad n = 1, 2, \cdots,$$

might possibly form a tannakian category. This (together with the generalized Ramanujan conjecture) would imply the existence of a locally compact group L_F, whose irreducible, unitary n-dimensional representations parametrize all of $\Pi_{\mathrm{cusp}}(GL_n)$. We shall assume that L_F exists in what follows. We shall also assume that there are injections $L_{F_v} \hookrightarrow L_F$, determined up to conjugacy, as well as a canonical surjective map $L_F \twoheadrightarrow W_F$ with compact connected kernel. (See [K.1, §12].) For us, the introduction of L_F is primarily for bookkeeping. The reader can pretend that L_F is the Weil group, or even the Galois group of a finite Galois extension E.

In [L.5, §2], Langlands also introduced the collection of *isobaric* representations, a subset of $\Pi_{\mathrm{aut}}(GL_n)$ which in turn contains $\Pi(GL_n)$. The significance of the isobaric representations is that they are in bijective correspondence with the (equivalence classes of) all semisimple representations of L_F of dimension n. In other words, they are parametrized by maps

$$w \longrightarrow \phi_1(w)\,|w|^{r_1} \oplus \phi_2(w)|w|^{r_2} \oplus \cdots \oplus \phi_m(w)|w|^{r_m}, \qquad w \in L_F,$$

where each $r_i \in \mathbb{R}$, ϕ_i is an irreducible unitary representation of L_F of dimension d_i, and $d_1 + \cdots + d_m = n$. What are the maps that correspond to the subset $\Pi_{\mathrm{disc}}(GL_n)$ of all the isobaric representations? They are just the maps with $\phi_1 = \phi_2 = \cdots = \phi_m = \phi$, and $r_1 = \frac{m-1}{2}$, $r_2 = \frac{m-3}{2}, \ldots, r_m = -\frac{m-1}{2}$. Now there is a nice way to characterize these particular maps within the set of all n-dimensional representations of L_F. Given an irreducible unitary representation ϕ of L_F of dimension d, with $n = md$, let ρ_m be the unique irreducible representation of $SL(2,\mathbb{C})$ of dimension m. Then

$$\psi(w,u) = \phi(w) \otimes \rho_m(u), \qquad w \in L_F,\ u \in SL(2,\mathbb{C}),$$

is an irreducible n-dimensional representation of $L_F \times SL(2,\mathbb{C})$. Having constructed ψ, we can define

$$\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}), \qquad w \in L_F.$$

In other words,

$$\phi_\psi(w) = \phi(w)|w|^{\frac{m-1}{2}} \oplus \cdots \oplus \phi(w)|w|^{-\frac{m-1}{2}}.$$

The maps corresponding to $\Pi_{\text{disc}}(GL_n)$ are thus obtained from irreducible representations of $L_F \times SL(2,\mathbb{C})$ whose restriction to L_F is unitary. In a similar vein, one sees that the maps corresponding to $\Pi(GL_n)$ are obtained by allowing the representations ψ to be reducible.

We return now to the case that G is arbitrary. Let $\Psi(G)$ denote the set of $\hat{G}$-orbits of admissible L-homomorphisms

$$\psi : L_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

such that the projection of $\psi(L_F)$ onto $\hat{G}$ is bounded. To any such ψ one can associate a finite group $\mathcal{S}_\psi$ [A.2, §8]. We will not reproduce the definition in general. However, if G is split over F, $\mathcal{S}_\psi$ equals $\pi_0(S_\psi/Z(\hat{G}))$, where $\pi_0(\ )$ denotes the group of connected components, S_ψ is the centralizer of $\psi(L_F \times SL(2,\mathbb{C}))$ in $\hat{G}$, and $Z(\hat{G})$ is the center of $\hat{G}$. One can also attach to ψ a certain sign character

$$\varepsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}.$$

Conjecture [A.2, §6, §8], [A.3, §4] For every $\psi \in \Psi(G)$ there exist

(i) finite local packets $\Pi_{\psi_v} \subset \Pi(G(F_v))$, which for almost all v contain precisely one representation with a $G(\mathfrak{o}_v)$-fixed vector; and
(ii) finite dimensional characters

$$s \longrightarrow \langle s,\pi\rangle = \prod_v \langle s,\pi_v\rangle, \qquad s \in \mathcal{S}_\psi,$$

defined for each representation in the global packet

$$\Pi_\psi = \{\pi = \bigotimes_v \pi_v \in \Pi(G(\mathbb{A})) : \pi_v \in \Pi_{\psi_v}\},$$

such that the multiplicity in $L^2(G(F)\setminus G(\mathbb{A}))$ of any representation $\pi \in \Pi(G(\mathbb{A}))$ equals

$$|\mathcal{S}_\psi|^{-1} \sum_{\{\psi\in\Psi(G):\pi\in\Pi_\psi\}} \sum_{s\in\mathcal{S}_\psi} \varepsilon_\psi(s)\langle s,\pi\rangle.$$

Remarks

1. The multiplicity formula implies that any representation $\pi \in \Pi(G)$ belongs to one of the packets Π_ψ . The nature of the parameters ψ then suggests that there is a Jordan decomposition for the elements in $\Pi(G)$ which is parallel to the Jordan decomposition for conjugacy classes in $G(F)$. For example, a parameter ψ can be called *unipotent* if the projection of $\psi(L_F)$ onto $\hat{G}$ equals $\{1\}$. The *unipotent automorphic representations* are then the constituents of sets $\Pi(G) \cap \Pi_\psi$, in which ψ is a unipotent parameter. The trivial one dimensional representation of $G(\mathbb{A})$ is the simplest example of such a representation. It corresponds to the principal unipotent class in $\hat{G}$. More interesting examples of unipotent automorphic representations have been constructed for (split) classical groups by Moeglin [Mg].

2. There are two conditions, which often hold for a given ψ, under which the multiplicity formula simplifies. Suppose that Π_ψ is disjoint from all the other packets $\Pi_{\psi'}$, $\psi' \neq \psi$, and that each of the functions
$$s \longrightarrow < s, \pi_v >, \qquad\qquad \pi_v \in \Pi_{\psi_v},$$
is a one dimensional abelian character. Then the multiplicity for any $\pi = \bigotimes_v \pi_v \in \Pi_\psi$ is 1 if the product $\prod_v < s, \pi_v >$ equals the sign character ε_ψ, and is 0 otherwise.

3. Suppose that G is quasi-split. Then
$$\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}), \qquad\qquad w \in L_F,$$
is an admissible L-homomorphism of L_F into ${}^LG$. The global packet Π_ψ should then contain the global Langlands packet Π_{ϕ_ψ}.

4. For maps ψ which are trivial on $SL(2, \mathbb{C})$, the conjecture contains nothing which is not already implicit in [L-L]. In this case the sign character ε_ψ is trivial. In general, however, ε_ψ is defined in terms of symplectic root numbers. Suppose that G is split. A parameter ψ determines a finite dimensional representation
$$R(s, w, u) \;=\; Ad(s\psi(w, u)), \qquad s \in S_\psi/Z(\hat{G}),\ w \in L_F,\ u \in SL(2, \mathbb{C}),$$
of $(S_\psi/Z(\hat{G})) \times L_F \times SL(2, \mathbb{C})$ on the Lie algebra of $\hat{G}$. Let
$$R \;=\; \bigoplus_{i \in I} (\lambda_i \otimes \mu_i \otimes \nu_i)$$

be the decomposition of R into irreducible representations. Observe that the determinant of an irreducible representation λ_i of $S_\psi/Z(\hat{G})$ can be regarded as a function on $\mathcal{S}_\psi = \pi_0(S_\psi/Z(\hat{G}))$. Observe also that for an irreducible representation μ_i of L_F, one can define the L-function $L(s,\mu_i)$ and root number $\varepsilon(s,\mu_i)$ from the embeddings $L_{F_v} \hookrightarrow L_F$. Let J be the subset of indices $i \in I$ such that the representation μ_i is symplectic, and such that $\varepsilon(\frac{1}{2},\mu_i) = -1$. Then

$$\varepsilon_\psi(s) = \prod_{i\in J} \det(\lambda_i(s)), \qquad s \in \mathcal{S}_\psi.$$

This formula for the sign character is strongly suggested by the spectral side of the trace formula. (See [A.3, especially §6].)

5. The parameters $\psi \in \Psi(G)$ have a direct bearing on our principal theme, automorphic L-functions, through the theory of Shimura varieties. They come up in the important problem of expressing the zeta function of a Shimura variety in terms of automorphic L-functions. Roughly speaking, the $SL(2,\mathbb{C})$ factor in a parameter ψ is the same object as the group $SL(2,\mathbb{C})$ that comes from the Lefschetz hyperplane section in cohomology. This allows one to determine the various degrees of cohomology to which a given ψ will contribute. (See [A.2, §9], [K.2, §8-10].) At the end of §10 of [K.2], Kottwitz states a conjectural decomposition of the λ-adic cohomology which implies a formula for the zeta function of a Shimura variety in terms of automorphic L-functions.

PART II

1 THE TRACE FORMULA, AND THE METHOD OF ZETA-INTEGRALS

We begin by taking a closer look at the two fundamental problems of Langlands' theory of automorphic L-functions.

Conjecture A Every general automorphic L-function

$$L(s,\pi,r) = \prod_{v\notin S} \det(1 - r(\sigma_v(\pi)q_v^{-s})^{-1} ,$$

initially defined as an Euler product convergent in some half-plane, continues to a meromorphic function in all of $\mathbb{C}$, with only finitely many poles, and a functional equation relating its values at s and $1-s$. (As remarked in Part I, a *precise* functional equation can be expected only after local factors $L(s,\pi_v,r)$ have been defined at the 'bad primes' v in S as well.)

Conjecture B Functoriality with respect to the L-group Suppose we are given reductive algebraic groups G and G' as in Section I.2, and a homomorphism

$$\rho : {}^L G \longrightarrow {}^L G' .$$

Then for each π in $\Pi_{aut}(G)$ there is a π' in $\Pi_{aut}(G)$ such that

$$\rho\{\sigma_v(\pi)\} = \{\sigma_v(\pi')\}$$

for all v outside some finite set $S \supset S_\infty$. In particular, for any $r : {}^L G' \rightarrow GL_d(\mathbb{C})$, this transfer of '$L$-packets' is such that

$$(*) \qquad L_S(s, \pi', r) = L_S(s, \pi, r o \rho).$$

Remarks

1. Suppose we take G arbitrary, $G' = GL_d$, ρ any L-group representation $\rho : {}^L G \rightarrow GL_d(\mathbb{C})$, and $r : {}^L G' \rightarrow GL_d(\mathbb{C})$ the standard representation (taking g to itself). Then $(*)$ above reads

   $$L_S(s, \pi', St) = L_S(s, \pi, \rho),$$

   with the left-hand side L-function a standard Godement–Jacquet L-function on GL_d. Thus Conjecture B indeed implies Conjecture A, and the immediate impression is that work on Conjecture A might be superfluous. This impression is misleading, however, since in practice it is often easier to establish Conjecture A directly, rather than appeal to the relevant form of Conjecture B.

2. Conjecture A has been successfully attacked in general using two different methods – the explicit construction of zeta-integrals, and the Langlands–Shahidi method using Eisenstein series (and their Fourier coefficients). A detailed survey of these methods – and their range of applicability – is the subject matter of [GeSh]. Our present state of knowledge concerning Conjecture A is roughly the following.
   (a) The existence of a meromorphic continuation is known for the L-function of almost any reductive quasi-split group and the 'standard' representation of its L-group;
   (b) except for $GL(n)$ (and its standard L-function), there are almost always non-trivial problems encountered in establishing an exact functional equation for $L(s, \pi, r)$, or the finiteness of the number of its poles; and
   (c) the best results have recently been obtained via the method of explicit zeta-integral representations – in fact, this method should ultimately prove most useful in number theory, since it makes possible

an analysis of the location (and possible arithmetic significance) of the poles of $L(s, \pi, r)$.

Examples illustrating (a)–(c) will be discussed at end of this lecture (Concluding Remarks and Theorems).

3. As we shall soon see, attempts to prove Conjecture A or B have drawn freely from *three* principal tools of the theory of automorphic forms: the trace formula, explicit zeta-integral representations of L-functions, and the theory of Θ-series liftings. We now proceed to discuss these topics in earnest.

Trace formula methods The prototype example here is the theory of base change already discussed in Part I. This example is also the most general to date, in that functorial lifting is proved for two quasi-split groups of rank n. Other examples involve either a 'compact' form of the trace formula (see [BDKV] and [Ro.1], which concern liftings between division algebras and GL_n), or else lower rank groups. See [La.3] for further discussion and references.

Unitary group examples ([Ro.2]) These examples concern functorial lifting from $U(2)$ to $U(3)$, and 'base change for $U(3)$'. Unlike the example of base change for $GL(n)$, one obtains here genuinely new information on the automorphic L-functions in question, as we now explain.

Let E denote a quadratic extension of the global field F, and V a three-dimensional Hermitian space which is defined over E and possesses an isotropic vector. Then set G equal to the unitary group $U(V)$ ('the' quasi-split unitary group in three variables), H the 'endoscopic' subgroup

$$U(2) \times U(1) = \left\{ \begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix} \text{ in } U(V) \right\},$$

and $\tilde{G}$ the group $Res_F^E G$. The corresponding L-groups are ${}^LG = GL_3(\mathbb{C}) \rtimes W_F$, where w acts via $g \to \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^tg^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$ if its image in $Gal(E/F)$ is non-trivial, ${}^LH = (GL_2(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes W_F$, with a similar action, and ${}^L\tilde{G} = (GL_3(\mathbb{C}) \times GL_3(\mathbb{C})) \rtimes W_F$, where w permutes the coordinates of ${}^L\tilde{G}^o$ (if the image of w in $Gal(E/F)$ is nontrivial). Finally, we consider the following L-group homomorphisms: let $\psi_G : {}^LG \to {}^L\tilde{G}$ denote the standard base-change imbedding $\psi_G(g, w) = (g, g, w)$, and let $\psi_H = {}^LH \to {}^LG$ extend the natural embedding of ${}^LH^o$ into ${}^LG^o$ via the formula

$$\psi_H(w) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \rtimes w$$

if the image of w in $Gal(E/F)$ is nontrivial, and otherwise,

$$\psi_H(w) = \begin{pmatrix} \mu(w) & & \\ & 1 & \\ & & \mu(w) \end{pmatrix} \rtimes w,$$

with μ a character of C_E whose restriction to C_F is $w_{E/F}$ (the quadratic character determined by E via class field theory).

Theorem ([Ro.2])

(a) There exist functorial transfers of L-packets corresponding to ψ_H and ψ_G.

(b) If π is cuspidal automorphic on $U(3)$, then its lift (through ψ_G) is again cuspidal (on $GL(3)$ over E) if and only if π is not the lift (through ψ_H) of any τ in $\Pi_{aut}(H)$.

Corollary For π any automorphic cuspidal representation of $U(V)$, and ξ any unitary idele-class character of E, set

$$L(s, \pi \otimes \xi) \equiv L(s, \psi_G(\pi) \otimes \xi),$$

the L-function on the right being the standard Godement–Jacquet L-function for $GL(3)$ over E of the lift of π. Then $L(s, \pi \otimes \xi)$ is always entire (for any ξ) if and only if π is not a ψ_H-transfer of some τ on H. Moreover, if π *is* of the form $\psi_H(\tau)$, then for some 'twist' ξ, $L(s, \pi \otimes \xi)$ has a pole on the line $Re(s) = 1$ if τ is cuspidal, and on $Re(s) = 3/2$ if τ is one-dimensional.

Remarks

1. The L-function $L(s, \pi \otimes \xi)$ may be interpreted as the 'standard' degree six L-function on $U(3)$. More precisely, let $G' = U(3) \times Res_F^E GL(1)$, so that

$${}^LG' = (GL_3(\mathbb{C}) \times \mathbb{C}^x \times \mathbb{C}^x) \rtimes W_F$$

with the obvious action of W_F on the connected component of ${}^LG'$. Let ρ denote the 6 dimensional representation of ${}^LG'$ induced from the 'standard' representation $St_3 \otimes St_1 \otimes 1$ of the (index two) normal subgroup

$GL_3(\mathbb{C}) \times \mathbb{C}^x \times \mathbb{C}^x$. Given π and ξ as above, we obtain the following conjugacy classes in ${}^LG'$ for each unramified v:

$$\sigma_v(\pi) \times \xi_w(\tilde{\omega}_w) \times \xi_{w'}(\tilde{\omega}_{w'}) \rtimes \sigma, \quad \text{if} \quad w, w'|v; \qquad \text{and}$$

$$\sigma_v(\pi) \times \xi_w(\tilde{\omega}_w) \times 1 \rtimes \sigma, \quad \text{if} \quad v \ \text{is inert.}$$

Thus the L-function $L(s, \pi \otimes \xi)$ introduced above is precisely the 'standard' L-function $L(s, \pi \otimes \xi, \rho)$ on G', all of whose analytic properties, as predicted by Conjecture A (including finiteness of poles and an exact functional equation), now follow directly from the Godement–Jacquet theory for $GL(n)$.

2. Our experience with $GL(n)$ does not prepare us for the existence of poles of automorphic L-functions to the right of the line $Re(s) = 1$ (the right-hand boundary of the critical strip). Indeed, it is a (non-trivial) fact that for π (resp. π') any automorphic cuspidal unitary representation of $GL_n(\mathbb{A})$ (resp. $GL_m(\mathbb{A})$), even the Euler product

$$\prod L(s, \pi_v \times \pi'_v)$$

defining $L_S(s, \pi \times \pi')$ *converges absolutely* for $Re(s) > 1$ (see [J-S], I). In particular $L_S(s, \pi \times \pi')$ has neither zeros *nor* poles in this half-plane, generalizing a classical result for Dirichlet's L-functions.

On the other hand, for groups other than $GL(n)$ – typically $Sp(n)$, we encounter a quite new phenomenon, namely the existence of cuspidal π whose standard L-functions can have poles at 3/2, 2, etc. The explanation for this fact is intimately tied up with the theory of theta-series liftings. This phenomenon is also related to the existence of counterexamples to the generalized Ramanujan conjecture, and the fact that many cuspidal π (outside $GL(n)$) fail to possess Whittaker models. In particular, for $U(3)$, those cuspidal π such that $L(s, \pi \otimes \xi)$ can have a pole to the right of $Re(s) = 1$ are non-tempered almost everywhere, possess no Whittaker model, and have the same L-function as an Eisenstein series (i.e., are CAP representations in the sense of [P-S.3]). All these matters will be discussed in detail at the end of the next section.

3. The discussion above suggests a question about $L(s, \pi \otimes \xi)$ on $U(3)$ left unanswered by the trace formula analysis of [Ro.2]. If π is such that its L-function can have a pole, can we characterize π by the peculiarity of its Fourier expansion? This question we shall discuss in earnest in §II.2 after first reviewing the use of explicit zeta-integrals in the theory of L-functions.

Zeta-integral methods Once again, the basic game plan is simple: it follows the lines introduced in Tate's analysis of the Dirichlet L-functions $L(s,\chi)$, and generalized by Godement–Jacquet for the standard $GL(n)$ L-functions $L(s,\pi)$.

For simplicity, we review the program for $GL(1)$ only. Given a grössen-character χ (an automorphic representation of $GL(1)$), we consider the global zeta-integral

$$\mathcal{Z}(s,f,\chi)=\int_{\mathbb{A}_F^x} f(x)\chi(x)|x|_{\mathbb{A}}^s d^x x,$$

where f is a Schwartz–Bruhat function on $\mathbb{A}$. The program we have in mind consists of the following five steps:

(1) Express the global zeta-integral as an Euler product of *local zeta-integrals* (in this case $\mathcal{Z}(s,f_v,\chi_v)$);
(2) Analyse the meromorphic behavior and functional equation of the *global* zeta-integrals;
(3) Do Step (2) for the *local* zeta-integrals as well;
(4) Interpret the *unramified* local zeta-integrals (in this case, meaning f_v is the characteristic function of the ring of v-adic integers, and χ_v is unramified) as an appropriate Langlands factor $[\det(I-r(\sigma_v(\pi))q_v^{-s})]^{-1}$ (in this case $1-\chi(v)q_v^{-s}$); and
(5) Establish additional basic properties of the local zeta-integrals, if possible introducing $L(s,\chi_v)$ *at the ramified primes* as a g.c.d. of the local zeta-integrals.

Because of Step (4), we say that $\mathcal{Z}(s,f,\chi)$ 'interpolates' $L(s,\chi)$, and we use all five steps of this 'L-function machine' to deduce the expected analytic properties of $L(s,\chi)$ from those of $\mathcal{Z}(s,f,\chi)$. For general groups, an account of how this method is carried out for $L(s,\pi,\rho)$ is found in [GeSh]; we content ourselves here only with some important examples which either motivate or are required in the discussion of Θ-series liftings in §II.2.

Hecke theory for $GL(n)$ Fix $\pi=\otimes\pi_v$ an irreducible unitary (not necessarily automorphic) representation of $GL(n,\mathbb{A}_F)$. For $GL(2)$, the fact that the L-function

$$L(s,\pi)=\prod_v L(s,\pi_v)$$

is 'nice' characterizes π as an automorphic cuspidal representation; more precisely the 'converse theorem' states that if every $L(s,\pi\otimes\chi)$ continues to an *entire* function in $\mathbb{C}$ which is bounded in vertical strips and satisfies the expected functional equation, then π is automorphic cuspidal. This result is proved using not a Godement–Jacquet zeta-integral, but rather a Hecke-type zeta-integral

$$\mathcal{Z}(s,\varphi,\chi) = \int_{\mathbb{Q}^x\backslash\mathbb{A}^x} \varphi_\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\chi(a)|a|^{s-\frac{1}{2}}d^x a$$

where φ_π belongs to the space of π. As is well-known, this result is useful in proving examples of Langlands functoriality. In particular, it implies that the Langlands reciprocity conjecture mentioned in Lecture §I.1 is true for two-dimensional Galois representations provided Artin's conjecture is true.

Throughout the 1970s, much effort was expended in obtaining converse theorems for $GL(n)$, $n > 2$. For $GL(3)$, it was found that a $GL(2)$-type result remains valid, and that it once again implies many interesting instances of Langlands functoriality. One such application is base change 'induction' for *non*-normal cubic extensions, proved in [J-P-S.2] and used by Tunnell in his proof of Artin's conjecture for arbitrary two-dimensional octahedral Galois representations. Another application of this theorem is (the proof of) Conjecture B for the symmetric square map $Ad: \ GL(2) \to GL(3)$. Complete trace formula proofs of these particular liftings have yet to be published.

For $n > 3$, converse theorems for $GL(n)$ apparently require twistings by automorphic representations of $GL(m)$, $m > 1$, i.e., the theory of generalized Rankin–Selberg convolutions discussed below.

The method of Rankin–Selberg, generalized The starting point is Jacquet's 1972 analysis of the zeta-integral interpolating the standard L-function $L(s,\pi_1 \times \pi_2,\rho)$ on $GL(2) \times GL(2)$ (i.e., $\rho: \ GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \to GL_4(\mathbb{C})$ is just the outer tensor product). His Rankin–Selberg type integral looks like

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi_1(g)\varphi_2(g)E(g,s)dg,$$

where $H = GL_2$ diagonally embedded in $G = GL_2 \times GL_2$, φ_i is in the space of π_i, and $E(g,s)$ is an Eisenstein series on GL_2. Subsequently, similar zeta-integrals were introduced by Jacquet, Piatetski-Shapiro and Shalika to interpolate the automorphic L-functions $L(s,\pi_1 \times \pi_2)$ on $GL(n) \times GL(m)$. As mentioned already in the first lecture, the results thus obtained for $L(s,\pi_1 \times \pi_2)$ yield a strong multiplicity one result for automorphic representations $\Pi(G)$ of $GL(n)$, and are used in the theory of base change for $GL(n)$. To be more precise, the results required are those describing if and when $L(s, \pi_1 \times \pi_2)$ has a pole on the line $Re(s) = 1$. Whereas explicit zeta-integrals are used to prove these results in [J-S](I), an alternate, shorter proof has recently been given in [W.1], using the Langlands–Shahidi theory of Eisenstein series.

Another possible application of these Rankin–Selberg convolutions is to the theory of converse theorems for $GL(n)$. The simplest type of theorem to state is the following: suppose Π is an irreducible unitary representation of $GL_n(\mathbb{A}_F)$ with the property that for *any* automorphic generic irreducible unitary representation τ of $GL_{n-1}(\mathbb{A}_F)$, the L-function $L(s, \Pi \times \tau)$ is 'nice', i.e., entire, bounded in vertical strips, and satisfies a functional equation of the form

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \hat{\pi} \times \hat{\tau}).$$

Then Π itself is automorphic cuspidal. In practice, one usually needs a converse theorem with weaker hypotheses, for example, $L(s, \pi \times \tau)$ need be 'nice' only for a restricted class of τ's (but then, as is well-known for the case $n = 2$, the conclusion is also weaker). These (and related) results are presently being developed by Piatetski-Shapiro and co-workers; see [P-S.1] and the discussion of functorial lifting below.

The general L-functions of Piatetski-Shapiro, Rallis, et al. In 1983–4, Piatetski-Shapiro and Rallis discovered a general type of zeta-integral which in one fell swoop generalized the Godement–Jacquet zeta-integral from $GL(n)$ to an arbitrary simple classical group, and paved the way for a vast generalization of the Rankin–Selberg method as well.

The idea is to consider zeta-integrals of the form

$$\text{(I)} \qquad \int_{H(F)\backslash H(\mathbb{A})} \varphi_G(h) E_H(h, s) dh$$

or

$$\text{(II)} \qquad \int_{G(F)\backslash G(\mathbb{A})} \varphi_G(g) E_H(g, s) dg,$$

where φ is a cusp form on some group G (belonging to an automorphic cuspidal representation π whose L-function we are trying to interpolate), and E is an Eisenstein series on some group H closely related to G. In both cases, the analytic properties of the relevant Eisenstein series must be established in order to obtain the meromorphic continuation of the corresponding automorphic L-functions. The key difference between Case I and II is that in the first case, $G \supset H$ (so that $\varphi(h)$ represents the restriction of φ to H), whereas in Case II, $G \subset H$; as we shall see in the few examples below, this distinction leads to different obstacles in trying to apply the L-function machine to these zeta-integrals.

Case I examples $H =$ *split* SO_{2n} (with an n-dimensional isotropic subspace), $G = SO_{2n+1} \supset H$, π is a *generic* cuspidal representation of $G(\mathbb{A})$ (i.e., each φ in π has nontrivial Fourier coefficients with respect to the standard maximal unipotent subgroup of G), and τ is a (generic) cuspidal representation of GL_n (regarded as a representation of the Levi component of the maximal parabolic subgroup P of H). Gelbart and Piatetski-Shapiro have applied several steps of the L-function machine to the zeta-integrals

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(h) E_\tau(h,s)dh \sim L(s, \pi \times \tau, r),$$

where $E_\tau(h,s)$ is an Eisenstein series on $H(\mathbb{A})$ attached to the parabolically induced representation $ind_{P(\mathbb{A})}^{H(\mathbb{A})} \tau|\det|^s$, and $L(s, \pi\times\tau, r)$ is the automorphic L-function on $G \times GL_n$ attached to the standard (tensor product) representation r of ${}^LG \times GL_n(\mathbb{C}) = Sp_n(\mathbb{C}) \times GL_n(\mathbb{C})$; see III.1.5 of [GeSh]

In a very interesting recent development, D. Ginsburg, Piatetski-Shapiro, D. Soudry, *et al.* have modified this method in order to cover the case of $G \times GL_k$, with k arbitrary relative to n; see [Gi] for the case $1 \leq k \leq n$. The significance of this general case is that it should *eventually* yield a functorial lifting from $G = SO_{2n+1}$ to GL_{2n} corresponding to the L-group homomorphism $\rho : {}^LG \to Sp_n(\mathbb{C}) \subset GL_{2n}(\mathbb{C})$; indeed, the converse theorem for GL_{2n} requires twistings $L(s, \pi \times \tau)$ with τ on GL_{2n-1}. Such an application of the theory of zeta-integrals to prove Conjecture B would be very exciting and should generalize to other classical groups. However, at present this work represents mostly a program for future research; much remains to be done, especially concerning the archimedean analysis in Steps 3 and 5 of the L-function machine, and the precise analytic properties of the global Eisenstein series. Moreover, even when complete, this work will give functoriality only for generic automorphic representations.

Case II examples: Rankin triple L-functions ([Ga] and [P-R.2]) Here $G = GL_2 \times GL_2 \times GL_2 \subset H = Sp_6$, so we are in Case II. Given a triplet of cuspidal representations of $GL(2)$, which we regard as a single representation of $GL(2) \times GL(2) \times GL(2)$, we derive the analytic properties of $L(s, \pi_1 \times \pi_2 \times \pi_3, \rho)$, where $\rho : {}^LG \to GL_8(\mathbb{C})$ is given by an outer tensor product, through the zeta-integrals

$$\int\limits_{G(F)\backslash G(\mathbb{A})} \varphi_1(g_1)\varphi_2(g_2)\varphi_3(g_3)E_H(g_1, g_2, g_3, s)dg_1dg_2dg_3 .$$

Here $E_H(h,s)$ is an Eisenstein series on H induced from the character $|\det|^s$ of the maximal parabolic subgroup P of H (whose Levi component is isomorphic to GL_3, and whose unipotent radical is abelian). The crucial point is that $P\backslash H$ has only one open orbit under the (right) action of $G \subset H$; this is the 'orbit yoga' which makes possible the appropriate factorization of these global zeta-integrals into local ones.

Other examples of zeta-integrals of type II (where $G \subset H$) also require an analysis of the orbit structure of $P\backslash H/G$ in order to obtain the necessary Euler product factorization (Step 1 of the L-function-machine). Two of the most striking applications of this method are the following:

Theorem [P-R.1] Given a simple classical group (such as SO_n, Sp_n), its standard L-function $L_S(s,\pi,St)$ has a meromorphic continuation and functional equation.

Theorem [PS-R-S] Suppose $G = G_2 \times GL(2)$, π (resp. τ) is an automorphic cuspidal *generic* representation of G_2 (resp. GL_2), and $\rho : G_2(\mathbb{C}) \times GL_2(\mathbb{C}) \to GL_{14}(\mathbb{C})$ is the standard representation of ${}^LG$ obtained from taking the tensor product of the standard embedding of $G_2(\mathbb{C}) \subset SO_7(\mathbb{C})$ in $GL_7(\mathbb{C})$ with the standard representation of $GL_2(\mathbb{C})$. Then $L_S(s, \pi \times \tau, \rho)$ has a meromorphic continuation and functional equation.

Concluding remarks

(a) The last example above is of special interest for the following reason. Groups like G_2 lie outside the range of applicability of the method of Langlands–Shahidi, since this method works only for groups which can be embedded as the Levi component of a parabolic subgroup of some larger reductive group. Although the method of explicit zeta-integrals has no such *a priori* limitations, it was nevertheless an open problem for years whether L-functions attached to G_2 could be analyzed via zeta-integrals.

(b) One of the main problems in the general theory of zeta-integrals comes from the difficulty in executing Step 5 of the L-function machine – i.e., in controlling the zeta-integrals at the 'bad' places. For example, in proving Theorem [P-R.1], one encounters an identity of the form

$$\int\limits_{G\times G(F)\backslash G\times G(\mathbb{A})} \varphi_1(g_1)\widetilde{\varphi_2}(g_2)E_H(g_1,g_2,s)dg_1dg_2$$

$$= \left(\prod_{v\in S} \mathcal{Z}_v(s,\varphi_1,\varphi_2,E)\right) L_S(s,\pi,St).$$

Now one does know exactly where the poles of the Eisenstein series E on the left side are located (see [P-R.1] and [P-R.2]). In general, however, one does not yet have complete control of the non-vanishing of the local zeta-integrals on the right-hand side of this identity; thus one can not conclude that the (finitely many?) poles of $L(s,\pi,\rho)$ are among those of the Eisenstein series. (See Lecture §II.2 for an alternate approach to this finiteness of poles result, at least for $G = Sp_{2n}$.) In the case of Rankin triple products, one *does* have sufficient control of the non-vanishing of the archimedean integrals, and hence precise information on the (finitely many) poles of $L(s,\pi_1\times\pi_2\times\pi_3)$. However, the rest of Step 5 – expressing the local integrals (at infinite primes) in terms of gamma factors – remains an open problem (except for the special case of holomorphic cusp forms; see [Ga]).

(c) Suppose we can prescribe exactly the location of the possible poles of a particular L-function. The method of zeta-integrals as described thus far still gives no information about characterizing those π for which these poles occur, nor does it give information about special values or non-vanishing properties of these L-functions. Such results seem to be accessible only via the theory of Θ-series liftings to be described in Section II.2 below. Typical of the results we wish to discuss are the following:

Theorem ([P-R.3]) Suppose $G = Sp(4)$, π is an automorphic cuspidal representation of G, and $L(s,\pi,\rho)$ is the degree 5 L-function associated to the standard embedding $\rho: SO_5(\mathbb{C}) \to GL_5(\mathbb{C})$. Then $L(s,\pi,\rho)$ is holomorphic for $Re(s) > 2$, has a meromorphic continuation to $\mathbb{C}$, and a simple pole at $s=2$ if and only if π is a certain Θ-series 'in two variables'.

Theorem ([W.2-3]) Suppose $G = PGL_2$, π is a cuspidal representation of $G(\mathbb{A})$ coming from a holomorphic cusp form f of even integral weight k, and $L(s,\pi)$ is the standard Hecke–Jacquet–Langlands L-function attached to π (so that $L(s,\pi)$ is entire in $\mathbb{C}$). Then $L(\frac{1}{2},\pi\otimes\chi_b) \neq 0$ for some quadratic character χ_b, if and only if f is the Shimura correspondence image of some cusp form $\tilde{f}$ of weight $\frac{k}{2}+\frac{1}{2}$ (in which case the non-vanishing of the χ_b-twisted L-function is related to the non-vanishing of the appropriate bth Fourier coefficient of $\tilde{f}$).

2 HOWE'S CORRESPONDENCE AND THE THEORY OF THETA-SERIES LIFTINGS

In his 1964 *Acta* paper, Weil gives a representation-theoretic formulation of the Siegel theory of theta-functions. In this theory, theta-functions comprise a space of functions on the so-called metaplectic group, functions which under right translation realize the metaplectic (oscillatory, or Weil) representation.

In more detail, first fix a local field F (not of characteristic 2), and a non-trivial additive character ψ of F. Let W denote a $2n$-dimensional symplectic vector space over F (equipped with antisymmetric form $< >$), and let $Sp(W)$ denote the symplectic group of W. Then Weil's metaplectic group $Mp(W)$ may be introduced as a (certain) group of unitary operators on some space S, fitting into the exact sequence

$$1 \longrightarrow T \longrightarrow Mp(W) \longrightarrow Sp(W) \longrightarrow 1,$$

with $T = \{z \in \mathbb{C} : \ |z| = 1\}$. Because the action of these operators on S depends on ψ, we should denote the metaplectic group by $Mp_\psi(W)$. We recall that:

(i) if $W = X \oplus X^v$ with X an n-dimensional isotropic ('Lagrangian') subspace of W, then S may be described as the (unitary completion of the) Schwartz–Bruhat space $S(X)$; moreover, the operators of Mp_ψ preserve $S(X)$;

(ii) $Mp_\psi(W)$ determines a *non-trivial* central extension of $Sp(W)$ by T, and hence does *not* produce an *ordinary* representation of $Sp(W)$; on the other hand (see [Rao]), there is a canonical splitting of $Mp_\psi(W)$ over the (unique) two-fold cover $\overline{Sp}$ of $Sp(W)$.

Similarly, given an $\mathbb{A}$-field F, a non-trivial additive character $\psi = \Pi\psi_v$ of $\mathbb{A}/F$, and a global symplectic space W, there is a metaplectic extension

$$1 \longrightarrow T \longrightarrow Mp_\psi(W_{\mathbb{A}}) \longrightarrow Sp(W_{\mathbb{A}}) \longrightarrow 1,$$

where $Mp_\psi(W_{\mathbb{A}})$ is now a group of unitary operators which preserves $\mathcal{S}(X_{\mathbb{A}}) = \otimes\mathcal{S}(X_v)$ and is compatible with the local metaplectic groups. The connection between this metaplectic group and the theory of automorphic forms derives from the fact that $Mp(W_{\mathbb{A}})$ splits (and then again canonically) over the subgroup of rational points $Sp(W)_F$ in $Sp(W_{\mathbb{A}})$. This splitting $r_F: Sp(W)_F \to Mp(W_{\mathbb{A}})$ is determined by the condition that for each γ in $Sp(W)_F$,

$$(*) \qquad \sum_{\xi \in X_F} (r_F(\gamma)\Phi)(\xi) = \sum_{\xi \in X_F} \Phi(\xi), \quad \forall\, \Phi \in \mathcal{S}(X_{\mathbb{A}}).$$

In particular, we can (and shall) regard $Sp(W)_F$ as a subgroup of $Mp_\psi(W_{\mathbb{A}})$, i.e., as a group of operators on $\mathcal{S}(X_{\mathbb{A}})$. If we let $\Theta: \mathcal{S}(X_{\mathbb{A}}) \to \mathbb{C}$ denote the functional $\Phi \to \Theta(\Phi) = \sum_{\xi \in X_F} \Phi(\xi)$, then $(*)$ simply says that this 'theta-functional' is $Sp(W)_F$-invariant.

Henceforth, we shall understand by 'Weil's metaplectic group' *either* $\overline{Sp}$ or the group of operators Mp_ψ. This latter group of operators determines an ordinary representation of $\overline{Sp}$ (or Mp_ψ itself) in the space S. We shall denote this representation by ω_ψ and refer to it as 'Weil's representation of the metaplectic group'.

By an automorphic form on the metaplectic group we understand a 'smooth' function on $Sp(W)_F \backslash Mp_\psi(\mathbb{A})$ or $Sp(W)_F \backslash \overline{Sp}(W_{\mathbb{A}}))$ satisfying the usual conditions of moderate growth, K-finiteness, etc. We refer the reader to [B-J] for the general definition of automorphic forms (which makes sense even for covering groups of algebraic groups). The theta-functional above gives an intertwining operator from the space of the Weil representation to the space of automorphic forms on $Mp_\psi(W_{\mathbb{A}})$ (or $\overline{Sp}$), namely

$$\begin{aligned} \Phi \to \Theta_\Phi(m_g) &= \Theta(m_g\, \Phi) \\ &= \sum_{\xi \in X_F} \omega_\psi(m_g)\Phi(\xi). \end{aligned}$$

The resulting automorphic forms are called theta-functions because, when $F = \mathbb{Q}$, $Sp(W) = SL_2$, and Φ is properly chosen, $\Theta_\Phi(g)$ essentially reduces to the classical theta-series

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}\,.$$

In general, $\Theta_\Phi(g)$ is a *genuine* function of $Mp_\psi(W)$, in the sense that

$$\Theta(\lambda g) = \lambda \Theta(g)$$

for all λ in T. Moreover, $\Theta_\Phi(g)$ still retains a basic distinguishing characteristic of classical theta-series, namely that most of the Fourier coefficients of Θ_Φ are zero. This fact plays a crucial role in the theory of L-functions (by way of facilitating the Shimura type zeta-integral constructions described below). However, to understand the full impact of the metaplectic representation in the theory of automorphic forms, one needs first to review Howe's correspondence and the theory of theta-series liftings.

Dual reductive pairs and Howe's correspondence ([Ho]) Howe's theory is a refinement of Weil's theory which converts the construction of automorphic forms via theta-functions into a machinery for lifting automorphic forms on one group to automorphic forms on another.

Definition A *dual reductive pair* in $Sp(W)$ is a pair of reductive subgroups G, H in $Sp(W)$ which comprise each other's centralizers in $Sp(W)$.

Examples

(I) $$G = Sp(W_1), \quad H = O(V_1) \subset Sp(W),$$

where $W = W_1 \otimes V_1$, W_1 is a symplectic space, V_1 is a quadratic space (with orthogonal group $O(V_1)$), and W is the symplectic space whose antisymmetric form is the tensor product of the forms on W_1 and V_1. Analogously, there are the Hermitian dual pairs $U(V_1), U(V_2) \subset Sp(V_1 \otimes V_2)$, where each V_i is a Hermitian space over some quadratic extension E of F, and the symplectic form on the Hermitian space $V_1 \otimes V_2$ is obtained by viewing $V_1 \otimes V_2$ as an F-vector space with $<\ ,\ >$ the 'imaginary' part of the form $(\ ,\ )_1(\ ,\ )_2$.

(II) $$G = GL(X_1), \quad H = GL(X_2) \subset Sp(W),$$

where $W = (X_1 \otimes X_2) \oplus (X_1 \otimes X_2)^*$ and

$$< (x, x^*), (y, y^*) >= y^*(x) - x^*(y).$$

Facts

(1) These examples exhaust the set of all *irreducible* dual reductive pairs; for a precise statement of the classification, see Chapter 1 of [MVW].

(2) Given a dual pair G, H in $Sp(W)$, the metaplectic extension $Mp_\psi(W)$ splits over G and H in all cases *except* when $G = Sp(W_1)$ is paired with an *odd*-dimensional orthogonal group $H = O(V_1)$, in which case $Mp_\psi(W)$ splits over H but *not* over G. This is a non-trivial fact whose proof is discussed in Chapter 3 of [MVW].

Philosophy for the duality correspondence Suppose G, H is a dual reductive pair in $Sp(W)$, and consider the *restriction* of the Weil representation ω_ψ of $Mp_\psi(W)$ to $G \times H$ (or rather, to a subgroup of $Mp_\psi(W)$ which is isomorphic to $G \times H$; we ignore the fact that this might not always be possible if $G = Sp(W_1)$). Because G and H are each others' mutual centralizers in $Sp(W)$, $\omega_\psi \big|_{G\times H}$ should decompose into irreducible representations of the form $\pi_1 \otimes \pi_2$, with π_2 an irreducible representation of H determined by the irreducible representation π_1 of G. In other words, but still roughly speaking, each π_1-isotypic component of $\omega_\psi \big|_{G\times H}$ should provide an irreducible $G \times H$-module of the form $\pi_1 \otimes \pi_2$, with $\pi_2 = \Theta_\psi(\pi_1)$ the 'Howe correspondence' image of π_1 on H; symbolically,

$$\omega_\psi \Big|_{G\times H} = \bigoplus_{\pi_1 \text{ in } G^\wedge} \pi_1 \otimes \Theta_\psi(\pi_1),$$

where the sum is over those π_1 in $G^\wedge$ which 'occur' in ω_ψ.

More precisely, we say that π_1 occurs in ω_ψ if $Hom_G(\omega_\psi, \pi_1) \neq \{0\}$, in which case we set

$$S(\pi_1) = \bigcap ker\ f\ , \quad \text{where } f \text{ runs through } Hom_G(\omega_\psi, \pi_1)$$

and

$$S[\pi_1] = S/S(\pi_1) \quad (\text{where } S \text{ is the space of } \omega_\psi).$$

The space $S(\pi_1)$ is G-stable (since each $ker\ f$ is), and H-stable (since H and G commute). By passage to the quotient, one obtains a representation of $G \times H$ in $S[\pi_1]$ which must be of the form $\pi_1 \otimes \pi'_2$ for some smooth (not necessarily irreducible) representation π'_2 of H. *Howe's conjectured duality correspondence* amounts to the assertion that there exists a unique irreducible quotient of π'_2, i.e., a unique invariant subspace of π'_2 whose quotient produces an *irreducible* representation π_2 of H. Assuming this quotient exists, we call π_2 the Howe image of π_1, and denote it by $\Theta_\psi(\pi_1)$.

Remark The Howe correspondence just described is symmetric in G and H, i.e., it doesn't matter which group we take as the 'domain' group. Thus the correspondence $\pi \to \Theta_\psi(\pi)$ goes in both directions!

From the work of Howe (see [W.4] and [MVW]), we have the following (local) result:

Theorem

(a) The Howe correspondence exists (at least for $p \neq 2$).

(b) If π_1 is unramified, so is $\Theta_\psi(\pi_1)$; in fact, whenever possible, this correspondence should be functorial with respect to the L-group, in a sense to be explained below, and in further examples.

Examples

1. (O_n, Sp_n) ([Li]). Consider the dual pair (Sp_n, O_n) where O_n is the orthogonal group of a non-degenerate quadratic form of dimension n, and n *is even.* Then Howe's duality correspondence gives rise to an injection from the unitary dual of O_n to the unitary dual of Sp_n. Assume that the quadratic form defining O_n is split and the character ψ defining Weil's representation is unramified. Then Howe's correspondence takes any unramified π in $O_n^\wedge$ to an unramified representation $\Theta_\psi(\pi)$ of $Sp_n(F)$; moreover, these representations are functorially related as follows: there is a natural map $\rho:\ {}^LO_n \rightarrow {}^LSp_n \approx SO_{2n+1}(\mathbb{C})$ such that the conjugacy class in ${}^LSp_n$ parametrizing $\Theta_\psi(\pi)$ is just $\rho(\sigma(\pi))$; in terms of local L-functions, this relation reads

$$L(s, \Theta_\psi(\pi)) = \zeta(s)\prod_{i=1}^{n/2}\zeta(s+i)\zeta(s-i)L(\pi, s),$$

a very special case of Theorem 6.1 of [Ra.1]. The injection $\Theta_\psi:\ O_n^\wedge \rightarrow Sp_n^\wedge$ is also a special case of 'explicit Howe duality in the stable range'; see [Ho.2], [Li.2] and [So.2] for more general results.

2. *Shalika–Tanaka Theory* ([S-T]). Let F be a local field, $H = SL_2(F) = Sp(F^2)$, where $F^2 = F \oplus F$ is equipped with the form $<(x,x'),(y,y')> = xy' - x'y$, and $G = SO(E)$, the (special) orthogonal group of the quadratic space E over F (so that $G = E^1 =$ the norm 1 group of E). Modulo the fact that G is not the full orthogonal group, (G, H) is a dual reductive pair in $Sp(W)$, where $W = F^2 \otimes E = X \oplus X^v$, and $X \approx F \otimes E \approx E$. In this case, ω_ψ acts in $L^2(E)$, $\omega_\psi \Big|_{G\times H}$ is an ordinary representation and $\omega_\psi|_G$ acts through the 'regular representation' of E^1 in $L^2(E)$. Thus every character χ in $G^\wedge$ 'occurs in ω_ψ' (in fact discretely). Moreover, if F is nonarchimedean, it turns out that $\pi_2 = \pi_2(\chi) = \Theta_\psi(\chi)$ is an irreducible supercuspidal representation of $H = SL_2(F)$, *unless* $\chi^2 = 1$.

The caveat $\chi^2 \neq 1$ is necessary here because we are dealing with $SO(2)$ in place of $O(2)$. Indeed, when χ is the unique character of order 2, $\Theta_\psi(\chi)$ is the sum of *two* irreducible supercuspidal representations π^+

and π^- and when $\chi = 1$, $\Theta_\psi(\chi)$ is an irreducible principal series representation (which is class 1 whenever E and ψ are unramified). Moreover, $\chi^2 \neq 1$ implies $\Theta_\psi(\chi) = \Theta_\psi(\chi^{-1})$; thus the correspondence $\chi \to \Theta_\psi(\chi)$ is two-to-one for such χ. On the other hand, *if we take* $O(2)$ *in place of* $SO(2)$, we get a one-to-one correspondence $\rho_\chi \leftrightarrow \Theta_\psi(\rho_\chi)$ between the irreducible (two-dimensional) representations of $O(2)$ (excluding the unique non-trivial character of $O(2)/SO(2)$) and certain *irreducible* representations $\Theta_\psi(\rho_\chi)$ of SL_2; namely, $\rho_\chi \to \Theta_\psi(\chi)$ when $\chi^2 \neq 1$ and $\rho_\chi = ind_{SO(2)}^{O(2)}\chi$, $\rho_\chi \to \pi^+$ or π^- when ρ_χ is a character of $O(2)$ non-trivial on $SO(2)$, and $\rho_\chi \to \theta_\psi(1)$ if ρ_χ is the trivial character.

Locally, the example $(SL_2, SO(2))$ is of interest because it provides an explicit construction of supercuspidal representations of $SL_2(F)$. Globally, it gives a generalization of the classical construction of cusp forms (both holomorphic and real-analytic) due to Hecke and Maass. More precisely, given a character $\chi = \Pi\chi_v$ of $E^1_{\mathbb{A}}$ trivial on E^x, we may consider the irreducible representation $\pi = \pi(\chi) = \otimes\pi_{\psi_v}(\chi_v)$ of $SL_2(\mathbb{A})$. (When E_v remains a field, $\pi(\chi_v)$ is as described above; otherwise, $\pi(\chi_v)$ is a principal series representation of $SL_2(F_v)$, unramified if χ_v is; in any case, $\pi(\chi_v)$ is class 1 for almost every v, and therefore $\pi(\chi)$ is a well-defined element of $\Pi(SL_2(\mathbb{A}))$. In fact, $\pi(\chi)$ is an automorphic representation of $SL_2(\mathbb{A})$, a result which Shalika and Tanaka establish by realizing $\pi(\chi)$ directly in the space of χ-isotypic theta-functions

$$f_\chi^\Phi(g) = \int_{E^x\backslash E^1_{\mathbb{A}}} \sum_{\xi\in E} \omega_\chi(tg)\Phi(\xi)\chi(t)d^xt.$$

This method generalizes, as we shall now see.

The global Howe correspondence Suppose $\pi_1 = \otimes\pi_v$ is an irreducible unitary representation of $G(\mathbb{A})$ which occurs in ω_ψ, i.e., for each fixed v, $Hom(\omega_\psi, \pi_v) \neq 0$. Then we can form the irreducible representation

$$\pi_2 = \Theta_\psi(\pi_1) = \otimes\Theta_\psi(\pi_v)$$

of $H(\mathbb{A})$, where for each v, $\Theta_\psi(\pi_v)$ is the local Howe image of π_v, and for almost every v, π_v and $\Theta_\psi(\pi_v)$ are class 1. We call π_2 the *Howe lift* of π_1.

Conjecture The irreducible representation $\pi_2 = \Theta_\psi(\pi_1)$ of $H(\mathbb{A})$ is (usually) automorphic if π_1 is. Moreover, the lift $\pi \to \Theta_\psi(\pi)$ should be functorial whenever possible.

Remarks

(1) This conjecture has deliberately been stated in a vague way, since not enough is known yet to justify making a more precise statement, and there are already some delicate counter examples, some of which will be described below.

(2) Although one might one day be able to establish the automorphy of $\Theta_\psi(\pi)$ in general using the trace formula, at present the best way to attack Howe's conjecture is by way of the theory of Θ-series liftings, generalizing what we described already in Example 2 above.

The theory of Θ-series liftings This theory makes it possible to prove the automorphy of $\Theta_\psi(\pi)$ by directly constructing a realization of $\Theta_\psi(\pi)$ inside the space of automorphic forms on $H(\mathbb{A})$. To simplify the exposition, let us suppose that π is an automorphic *cuspidal* representation of $G(\mathbb{A})$ and H_π is an irreducible subspace of $L_0^2(G(F)\backslash G(\mathbb{A}))$ realizing π. Then we can consider a space of functions on $H(\mathbb{A})$ given by the integrals

$$f_\varphi(h) = \int_{G(F)\backslash G(\mathbb{A})} \Theta_\Phi^\psi(g,h)\varphi(g)dg \tag{2.1}$$

where $\varphi \in H_\pi$ and $\Theta_\Phi^\psi(g,h)$ is the restriction to $G(\mathbb{A}) \times H(\mathbb{A})$ of any theta-function Θ_Φ on $Mp(W)_\mathbb{A}$. Because Θ_Φ^ψ is known from Weil's theory to be a slowly-increasing continuous function on $H \times G(F)\backslash H \times G(\mathbb{A})$, and because φ is cuspidal (and hence rapidly decreasing) on $G(F)\backslash G(\mathbb{A})$, it follows that each of these integrals converges absolutely, and defines an automorphic form on $H(F)\backslash H(\mathbb{A})$. Let us denote by $\Theta(\pi,\psi)$ the space of functions f_φ^Φ so generated on $H(F)\backslash H(\mathbb{A})$.

What is the relation between this new $H(\mathbb{A})$-module $\Theta(\pi,\psi)$ (*the theta-series lifting of* π), and the $H(\mathbb{A})$-module $\Theta_\psi(\pi)$ (the Howe lifting of π)? The answer should be that $\Theta(\pi,\psi)$ realizes $\Theta_\psi(\pi)$ *provided* $\Theta(\pi,\psi)$ is not identically zero! The problem is that the non-vanishing of $\Theta(\pi,\psi)$ is subtle to detect.

Prototype example ([W.2]) Fix $G = SL(2)$, and $H = SO(V)$, where V is the 3-dimensional space of trace zero 2×2 matrices equipped with the quadratic form $q(X) = -\det(X)$. In this case, Howe's correspondence relates representations of $SO(V)$ (which is isomorphic to $PGL(2)$) with representations of $SL(2)$ (or rather the two-fold cover of $SL(2)$, since $M_{p_\psi}(F^2 \otimes V)$ does not split over $SL(2)$). In particular, the local correspondence establishes a bijection between the set of irreducible admissible *genuine* representations of $\overline{SL(2)}$ which possess a ψ-Whittaker model and the set of all irreducible admissible representations of $PGL(2)$. Globally, we are dealing with an integral of the form (2.1), where G is $SL(2)$; although both $\Theta(g,h)$ and $\varphi(g)$ are now *genuine* functions on the metaplectic cover of $SL(2)$, their *product* is naturally defined on $SL(2)$.

Now consider the question whether π automorphic on $H(\mathbb{A})$, say, implies $\Theta_\psi(\pi)$ automorphic on $G(\mathbb{A})$, and if so, is $\Theta_\psi(\pi)$ realizable in $\Theta(\pi,\psi)$? The answer (in the direction $PGL_2 \to \overline{SL_2}$) is that $\Theta_\psi(\pi)$ is automorphic *if and only if* $\varepsilon(\pi,\frac{1}{2}) = 1$; here $\varepsilon(\pi,s)$ is the ε-factor in the functional equation

$$L(s,\pi) = \varepsilon(\pi,s)L(1-s,\pi)$$

satisfied by the Hecke–Jacquet–Langlands L-function of π. Moreover, even when the condition $\varepsilon(\pi,\frac{1}{2}) = 1$ is satisfied, $\Theta_\psi(\pi)$ will be realizable in $\Theta_\psi(\pi,\psi)$ only if this latter space is non-zero; this happens if and only if the stronger condition

$$L(\frac{1}{2},\pi) \neq 0$$

is satisfied.

In the reverse direction, $\overline{SL_2} \to PGL_2$, Howe's correspondence is equally subtle. Here it turns out that $\Theta_\psi(\sigma)$ is always automorphic on PGL_2 if σ is automorphic, but $\Theta(\sigma,\psi)$ is non-zero (and then realizes $\Theta_\psi(\sigma)$) if and only if the ψth Fourier coefficients

$$f_\sigma^\psi(g) = \int f_\sigma\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\overline{\psi(x)}dx$$

do not vanish identically for f_σ in the space of σ. There is also an intriguing characterization of the non-vanishing of these Fourier coefficients in terms of special values of L-functions; this we shall describe later after introducing some general zeta-integrals involving theta-series (zeta-integrals of Shimura type).

Examples involving functoriality The construction of Shalika–Tanaka relating χ and $\pi(\chi)$ was independently obtained in [J-L] with E and $GL(2)$ in place of E^1 and $SL(2)$. At this level, it is easy to check that the lifting $\chi \to \pi(\chi)$ is 'functorial' with respect to the natural L-group morphism Ψ : $(\mathbb{C}^x \times \mathbb{C}^x) \rtimes W_F \to GL_2(\mathbb{C}) \times W_F$. Another example involving functoriality is the following.

Let $G = GSp_2$, the group of symplectic similitudes of a 4-dimensional symplectic space W_1, and let $GSO(V_{3,3})$ denote the group of orthogonal similitudes of a 6-dimensional orthogonal space $V_{3,3} = \Lambda^2(F^4)$ equipped with the inner product

$$(W, W') = W \wedge W' \in \Lambda^4(V) \approx F.$$

Although we are dealing here with groups of similitudes inside $GSp(W_1 \otimes V_{3,3})$, a suitable modification of the theory of dual reductive pairs leads us to liftings $\pi \leftrightarrow \Theta_\psi(\pi)$ and $\pi \leftrightarrow \Theta(\pi, \psi)$ between GSp_2 and $GSO(V)$.

Remarks

(1) When dealing with groups of *similitudes*, it turns out that Howe's lifting (or the theta-series lifting) is *independent* of ψ. (The subscript ψ may therefore be suppressed in these cases, though we shall usually refrain from doing so.)

(2) There is a natural injection of $GL_4/\mathbb{Z}_2$ into $GSO(V)$. Therefore, the Θ-series correspondence $\pi \to \Theta(\pi, \psi)$ may be viewed as a correspondence between representations of GSp_2 and GL_4 (by restricting functions in $\Theta(\pi, \psi)$ to $GL_4/\mathbb{Z}_2$); similarly, Howe's correspondence for the pair $(GSp_2, GSO(6))$ naturally defines a correspondence between representations of GSp_2 and $H = GL_4$ which we again denote by $\Theta_\psi(\pi)$. With these remarks in mind, we state the following:

Theorem ([J-P-S.3])

(a) Suppose π is an automorphic cuspidal representation of $G(\mathbb{A})$; then $\Theta(\pi, \psi) \not\equiv \{0\}$ *if and only if π is globally generic*, i.e., possesses a standard Whittaker model (on the space of its ψ-Fourier coefficients). In this case, $\Theta(\pi, \psi)$ realizes the Howe lift $\Theta_\psi(\pi)$, and the lifting is compatible with Langlands' functoriality in the following sense: if $\rho : {}^LG \to {}^LH = GL_4(\mathbb{C})$ denotes the standard embedding of $GSp_2(\mathbb{C})$ in $GL_4(\mathbb{C})$, and $\{\sigma_v(\pi)\}$ is the collection of conjugacy classes in ${}^LH$ determined by $\pi = \otimes \pi_v$, then $\rho\{\sigma_v(\pi)\}$ coincides with the collection of conjugacy classes in $GL_4(\mathbb{C})$ determined by $\Theta_\psi(\pi)$.

(b) Suppose Π is an automorphic cuspidal representation of $H(\mathbb{A})$. Then Π is the Θ-series lift of some (globally generic) π on $G(\mathbb{A})$ as in (a) if and only if the degree 6 L-function

$$L(s, \Pi \otimes \chi, \Lambda^2)$$

has a pole at $s = 1$ for some grössen-character χ.

Concluding remarks

1. The example just given shows how Conjecture B can be proved using the theory of Θ-series liftings, at least for generic π. One should also be able to establish this lifting (again for generic π) as a special case of the 'converse theorem' program outlined in Section I.1; for the case at hand, this requires an analysis of the L-functions $L(s, \Pi \times \tau)$ on $GSp_2 \times GL_2$ already studied in [PS-So]. In general, for arbitrary π, one must also eventually be able to establish this lifting using the trace formula. In any case, the functorial identity
$$L(s, \Pi, \Lambda^2) = L(s, \pi, \rho o \Lambda^2),$$
where $\rho : {}^L G \to {}^L H$, already implies that (some twisting of) $L(s, \Pi, \Lambda^2)$ must have a pole at $s = 1$ if $\Pi = \Theta_\psi(\pi) = \rho(\pi)$, since $\rho o \Lambda^2$ contains a one-dimensional subrepresentation (and therefore $L(s, \pi \otimes \chi, \rho o \Lambda^2)$ will – for some χ – contain the Riemann zeta-function as a factor).

2. As the preceding examples confirm, instances of functorial lifting can be established using the trace formula, L-functions, the theory of theta-series liftings, or any combination thereof. In some cases, such as the Shalika–Tanaka example $\chi \to \pi(\chi)$, each one of these methods provides an (alternate) proof; the L-function method was applied in §12 of [J-L] (albeit at the level of $GL(2)$), whereas the trace formula approach was developed in [L-L], and led to new and provocative results.

Zeta-integrals of Shimura type Although of interest in its own right, our lengthy detour through the theory of Θ-series was motivated entirely by our interest in the analytic properties of automorphic L-functions. The connection between these subjects is provided by zeta-integrals of Shimura type, themselves modifications of Rankin–Selberg integrals involving Θ-series.

To explain the general construction, we fix $G = Sp_n$, and we consider zeta-integrals of the type

$$\zeta(s, \varphi, \Phi, F) = \int_{G(F)\backslash G(\mathbb{A})} \varphi(g) \Theta_T^\Phi(g) E(g, F, s) dg. \tag{2.2}$$

Here $\varphi(g)$ is a cusp form in the subspace of $L_0^2(G(F)/G(\mathbb{A}))$ realizing the irreducible cuspidal representation π of $G(\mathbb{A})$; T is an $n \times n$ symmetric non-degenerate matrix, which we confuse with the n-dimensional orthogonal space V_T it determines; $\Theta_T^\Phi(g)$ is (the value at $(g, h) = (g, 1)$ of) the theta-kernel
$$\Theta_T^\Phi(g, h) = \sum_{\xi \in M_{n,n}(F)} (\omega_\psi(g, h)\Phi)(\xi)$$

corresponding to the dual pair $(Sp_n, O(V_T))$ in Sp_n and the choice of Schwartz–Bruhat function Φ on the Lagrangian subspace $F^n \otimes V_T$ (viewed as $n \times n$ matrix space); finally, $E(g, F, s)$ is the Eisenstein series on $G(\mathbb{A})$ of the form

$$\sum_{\gamma \in P(F)\backslash G(F)} F_s(\gamma g)$$

where $P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$ in G is the parabolic subgroup whose Levi component M is isomorphic to GL_n, and $F_s(g)$ belongs to $Ind_{P(\mathbb{A})}^{G(\mathbb{A})} |\det_M m(p)|^{s+\frac{n+1}{2}}$. For convenience, *we assume n is even* (and therefore the metaplectic group never need occur here).

The zeta-integral $\zeta(s, \varphi, \Phi, F)$ interpolates the standard L-function of degree $2n+1$ for $G(\mathbb{A})$, and the desired properties of this L-function are thus obtained by an application of the L-function machine to the above zeta-integrals. For example, we have the following result, first discussed at the end of the last Lecture.

Theorem [P-R.3] Suppose π is an automorphic cuspidal (not necessarily generic) representation of $Sp_2(\mathbb{A})$, and suppose T is a symmetric invertible 2×2 matrix in $M_{2,2}(F)$ such that for some φ in the space of π, the 'T-Fourier coefficient'

$$\varphi_T(g) = \int_{S(F)\backslash S(\mathbb{A})} \varphi\Big(\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} g\Big) \overline{\psi(tr\ T\chi)} dX$$

does not vanish (there $S = \{X\}$ denotes the F-vector space consisting of symmetric 2×2 matrices). Let ρ denote the standard 5-dimensional representation of the L-group of $SO_5(\mathbb{C})$ of Sp_2 and let $\chi_T(a)$ denote the quadratic character $(a, -\det T)$. Then the L-function

$$L_S(s, \pi \otimes \chi_T, \rho), \qquad Re(s) \gg 0,$$

has meromorphic continuation to $\mathbb{C}$, with only finitely many poles, and is analytic for $Re(s) > 2$.

Sketch of proof Everything follows from the basic identity

$$\zeta(s, \varphi, \Phi, F) = G_\infty(s) L_S(s + \frac{1}{2}, \pi \otimes \chi_T, \rho), \tag{2.3}$$

where $F_s(g)$ is suitably normalized, i.e., multiplied by a suitable product of zeta-functions, such that (1) the identity (2.3) holds *and* (2) the resulting (normalized) Eisenstein series has only finitely many poles (in this case, at certain half-integral or integral points of the interval $[-\frac{3}{2},\frac{3}{2}]$). The function $G_\infty(s)$, which depends on the local data Φ_v and $F_{v,s}$, is a meromorphic function in $\mathbb{C}$ which can be chosen to be non-zero at any one of the possible poles of $E(s,g,F)$. From this it is clear that $L(s+\frac{1}{2},\pi\otimes\chi_T,\rho)$ can have a pole at $s=s_0$ only if $E(s,g,F)$ does. Moreover, a pole of $E(s,g,F)$ will produce a pole of $L(s',\pi\otimes\chi_T,\rho)$ only if this pole survives integration against $\varphi(g)\Theta(g)$. This circumstance brings us full circle back to the theory of Θ-series liftings. In particular, we have the following important Corollaries to the proof just sketched:

Theorem [Li] The L-function $L_S(s,\pi\otimes\chi_T,\rho)$ has a pole at $s=2$ if and only if the Fourier expansion of any φ in π has only one 'orbit' of non-zero Fourier coefficients φ_T, i.e., π is T-distinguished in the sense of [Li].

Remark There is always *some* nondegenerate T such that $\varphi_T\neq 0$; this was proved earlier by Li in his thesis [Li.3]. But then for $T' = {}^tA^{-1}TA^{-1}$, with A in GL_n, we have

$$\varphi_{T'}(g)=\varphi_T\left(\begin{bmatrix} A^{-1} & 0 \\ 0 & {}^tA \end{bmatrix} g\right). \tag{2.4}$$

Thus $\varphi_{T'}\neq 0$ for all T' in the 'orbit' of T. The meaning of π being distinguished is that there is a T such that $\varphi_{T'}\neq 0$ if and only if T' is in the orbit of T. Li's characterization of distinguished π follows from (the analog for Sp_2 of) the first half of the following result.

Theorem [P-R.3] and [So] Consider the similitude group $G=GSp_2$ in place of Sp_2 (so now ${}^LG=GSp_2(\mathbb{C})$, with five dimensional representation ${}^LG\to PGSp_2(\mathbb{C})\approx SO_5(\mathbb{C})\subset GL_5(\mathbb{C})$). Then $L_S(s,\pi\otimes\chi_T,\rho)$ has a simple pole at $s=2$ if and only if either one of the following conditions are satisfied: (1) the theta-series lifting of π to $GO(T)$ is non-zero, i.e., $\Theta(\pi,\psi)\neq 0$; or (2) π is a CAP representation with respect to either the Borel subgroup of G, or the parabolic subgroup Q which preserves a line.

Remarks

1. The first condition follows immediately from the fact that $Res_{s=3/2}E(g,s,F)$ is independent of g, and therefore $L(s',\pi\otimes\chi_T,\rho)$ has a pole at $s'=2$ iff

$$Res_{s=3/2}\zeta(s,\varphi,\Phi,F)=c\int_{G(F)\backslash G(\mathbb{A})}\varphi(g)\Theta(g,1)dg\neq 0$$

i.e., iff $\Theta(\pi, \psi)$ is not identically zero on $GO(T)$. (Again, some modification of the theory of dual pairs is required for the similitude groups $GO(T)$ and GSp_2; in this case, as already mentioned in a recent remark, $\Theta(\pi, \psi)$ no longer depends on ψ.)

2. We shall come back to CAP representations at the end of Lecture 4. Suffice it now to give the definition: a cuspidal π is C(uspidal), A(ssociated to a) P(arabolic) if there exists a proper parabolic subgroup $P = MU$, and a cuspidal automorphic representation τ of $M(\mathbb{A})$, such that for almost all v, π_v is a constituent of $ind_{P_v}\tau_v$. Put more colorfully, π is in the 'shadow' of an Eisenstein series!

3. A similar characterization of those π such that $L(s, \pi \otimes \chi_T, \rho)$ might have a pole at a different singularity of $E(s, g, F)$ depends (at least) on the possibility of identifying the remaining residues of $E(s, g, F)$ via some kind of Siegel–Weil formula. For example, for the point $s = \frac{1}{2}$, such a Siegel–Weil formula is developed in [Ku-Ra-So], and applied to give a characterization of the pole of $L(s, \pi \otimes \chi_T, \rho)$ at $s = 1$ (analogous to the theorem above).

Comments on the Shimura zeta-integral

(1) Theorem [P-R.3] above (and also Li's theorem) generalizes to Sp_n with n *even*. It gives the finiteness of poles result missing from the authors' earlier work on $L(s, \pi, \rho)$ via the Godement–Jacquet type integral of [PR.1]. (Recall that for the latter zeta-integral, the required non-vanishing of the bad local integrals has yet to be established in full generality.) This result also improves on Shahidi's theory, which gives the finiteness of poles result only for *generic* π.

(2) The analysis of the poles of the Eisenstein series $E(g, s, F)$ is a delicate business involving intertwining operators for the induced representation $Ind|\det|^{s'}$; this is the subject matter of §4 of [P-R.2] as well as §3 of [P-R.1]. A still more difficult problem is the analogous analysis of intertwining operators and Eisenstein series for the induced (from cuspidal!) representations $Ind_p^G \tau |\det|^{s'}$ on, say, $G = SO_{2n+1}$; until this problem is resolved, the finiteness of poles result for the Rankin–Selberg L-functions on $G \times GL(n)$ will remain incomplete. For a discussion of what must be proved, see Chapters II and III of [Ge-Sh].

(3) For Sp_n, with n *odd*, these methods must be modified to involve Eisenstein series on the metaplectic group. Indeed $\Theta_T(g)$ is now a genuine function on the metaplectic cover of Sp_n, and hence must be multiplied by an Eisenstein series of the same 'genuine' type (so that the resulting integrand in the Shimura integral (2.2) will be naturally defined on Sp_n). For arbitrary n, the theory has not yet been worked out. However, for $n = 1$ we encounter the zeta-integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbb{A})} \varphi(g)\Theta^{\Phi}(g)E(g,F,s)dg$$

interpolating the degree 3 L-function for $SL(2)$ corresponding to the L-group homomorphism

$$\rho = Ad: \ PGL_2(\mathbb{C}) \longrightarrow SO_{2,1}(\mathbb{C}) \subset GL_3(\mathbb{C}).$$

In this case, Gelbart and Jacquet have shown that $L(s, \pi \otimes \chi, Ad)$ is entire for all twists χ, unless the theta-series lifting of π to an isotropic $G' = SO(2)$ is non-trivial. If this is so, there can be a pole at $s = 1$, again for reasons of Langlands functoriality. Indeed, such a π is then the Shalika–Tanaka lift of some π' on $SO(2)$ coming from an L-group homomorphism $\Psi: \ {}^LG' \rightarrow PGL_2(\mathbb{C})$ with the property that $\Psi o \rho$ contains the identity representation, i.e., $L(s,\pi,\rho) = L(s,\pi',\rho o\Psi)$ has a degree 1 factor producing a pole at $s = 1$. In any event, an application of the converse theorem for $GL(3)$ yields the Gelbart–Jacquet lifting from $GL(2)$ to $GL(3)$.

(4) A further modification of the zeta-integral (2.2) comes from replacing φ by an automorphic cusp form of half-integral weight, i.e., an automorphic cusp form on the metaplectic group. In case $n = 1$, this reduces to precisely the work [Shim] which got this entire business started, and which ties in with the work of Waldspurger discussed below. For general n, the theory is not yet developed.

(5) The method of the Shimura zeta-integral works for L-functions on groups other than Sp_n, for example $Sp_n \times GL_n$ and $U(3)$ ([GeRo]), where it gives results completely analogous to Theorem [P-R.3]. The latter results for $U(3)$ complement those which can be obtained via the trace formula, since they give intrinsic characterizations in terms of Θ-series liftings and Fourier expansions – of those π whose (twisted) degree six L-functions are not always entire.

Waldspurger's work (on the non-vanishing of Fourier coefficients and special values of L-functions) and counterexamples to Ramanujan's Conjecture Consider again the dual pair $(\overline{SL}_2, PGL_2)$. Recall that in this case, the theory of theta-series liftings gives a bijection between cuspidal σ on $\overline{SL}_2$ with non-vanishing ψth Fourier coefficients and cuspidal π on PGL_2 with $L(\pi, \frac{1}{2}) \neq 0$. But the question remains: how can we characterize the non-vanishing of this Fourier coefficient intrinsically in terms of σ? Waldspurger's answer comes from combining a Siegel–Weil formula with the theory of the Shimura integral.

Given any automorphic cuspidal representation σ on $\overline{SL}_2$, define its *Shimura image* on PGL_2 by

$$\pi = Sh_\psi(\sigma) \equiv \Theta(\sigma, \psi_a) \otimes \chi_a \, .$$

Here $\psi^a(x) = \psi(ax)$ is any character such that $\Theta(\sigma, \psi^a) \not\equiv \{0\}$, and χ_a is the corresponding quadratic character $x \to (a, x)$. In [W.2] it is shown that $Sh_\psi(\sigma)$ depends only on ψ, and not on the choice of a in F^x. From the bijective properties of the correspondence $\Theta(\cdot, \psi)$ recalled above, it then also follows that, for any automorphic cuspidal representation π of PGL_2,

$$\begin{gathered} \pi = Sh_\psi(\sigma) \text{ for some cuspidal } \sigma \text{ on } \overline{SL}_2 \\ \text{if and only if} \\ L(\tfrac{1}{2}, \pi \otimes \chi_a) \neq 0 \text{ for some } a \text{ in } F^x. \end{gathered}$$

It is interesting to note that this same characterization of the image of Sh_ψ has been sketched recently by Jacquet using the 'relative' trace formula (see [Ja]). However, the following remarkable result seems to lie deeper, and is (thus far) proved only in [W.3].

Theorem Suppose $\sigma = \otimes\sigma_v$ is an automorphic cuspidal representation of $\overline{SL}_2$, and f is any cusp form in the space of σ. Then f admits a non-zero ψth Fourier coefficient if and only if:

(i) for all v, σ_v has a ψ_v-Whittaker model; and

(ii) $L_S(\frac{1}{2}, Sh_\psi(\sigma)) \neq 0$.

Sketch of proof The only if direction is easy, since the non-vanishing of the ψ-Fourier coefficient immediately implies both the existence of a global (and hence local) Whittaker model, and the non-vanishing of the theta-series lift $\Theta(\sigma, \psi) = \pi = Sh_\psi(\sigma)$ (hence $L(\frac{1}{2}, \pi) = L(\frac{1}{2}, Sh_\psi(\sigma)) \neq 0$).

So suppose now that σ satisfies (i) and (ii). By the properties of the bijection $\sigma \leftrightarrow \Theta(\sigma, \psi)$, it will suffice to show that $\Theta(\sigma, \psi) \neq \{0\}$ (since this is equivalent to the non-vanishing of the ψth Fourier coefficients of f in σ); i.e., we must prove that for some choice of φ, Φ, and g,

$$\zeta(g) \equiv \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \varphi(h)\Theta_\psi^\Phi(g,h)dh \neq 0 \qquad \text{(see (2.1))}.$$

Note that for any $\varphi \neq 0$, there is some a in F^x so that $\varphi_{\psi^a} \neq 0$. If $a \in (F^x)^2$, then the fact that φ_ψ and $\varphi_{\psi^\lambda a}$ are related in an elementary fashion (see (2.4)) implies that we are done, i.e., $\varphi_\psi \neq 0$. Thus we may assume $a \notin (F^x)^2$. In this case, recall that X is the three-dimensional quadratic space on which $PGL_2 \approx SO(2,1)$ acts, fix $x_a = \begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix}$ in X, and decompose X into the line generated by x_a and the orthocomplement X'_a. We may take Φ of the form $\phi(\lambda x_a + x') = \phi_1(\lambda x_a)\phi_2(x')(x' \in X'_a)$, so that for t in the stabilizer T of x_a in PGL_2, we have

$$\zeta(t) = \int \varphi(h)\Theta_\psi^{\Phi_1}(h)\Theta^{\Phi_2}(t,h)dh$$

i.e., for $(t,h) \in T \times SL_2$, the Weil representation (or theta-kernel) decomposes in this simple way.

Let us now assume $\zeta(t) \equiv 0$ and show this leads to a contradiction of our hypotheses. Let $K = F(\sqrt{a})$, and view T as the anisotropic form of norm 1 elements of K^x. Since $T_F\backslash T_{\mathbb{A}}$ is compact, we can integrate the integral expression for $\zeta(t)$ with respect to $T(F)\backslash T(\mathbb{A})$, and interchange the order of integration to obtain

$$0 \equiv \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \varphi(h)\Theta_\psi^{\Phi_1}(h)\left(\int_{T(F)\backslash T(\mathbb{A})} \Theta^{\Phi_2}(t,h)dt\right)dh.$$

The Siegel–Weil type formula proved by Waldspurger asserts that the integral in parentheses above equals the value at $s = \frac{1}{2}$ of the Eisenstein series

$$E^{\phi_2}(h,s) = \sum_{\gamma \in B(F)\backslash SL_2(F)} f_s(\gamma h).$$

(Here $f_s(h) = L(s+\frac{1}{2},\chi_a)|\alpha_h|^{s-\frac{1}{2}}(\omega_\psi^2(h)\Phi_2)(0)$, where ω_ψ^2 denotes the Weil representation associated to the dual pair $SO(K) \times SL_2 \subset Sp_4$, and h in SL_2 has the Iwasawa decomposition $h = \begin{pmatrix} \alpha_h & * \\ 0 & \alpha_h^{-1} \end{pmatrix} n$, with $u \in K_{\mathbb{A}}$.) Since f_s belongs to $Ind_{B(\mathbb{A})}^{SL_2(\mathbb{A})}|\alpha_h|^s\chi_a$, and its value at $h = e$ is essentially $L(s+\frac{1}{2},\chi_a)$, $E^{\phi_2}(h,s)$ is a familiar *normalized Eisenstein series* on SL_2. For s

sufficiently large, this Eisenstein series converges absolutely, and hence we can write

$$0 = \operatorname*{value}_{s=\frac{1}{2}} \zeta^*(s), \qquad \text{where}$$

$$\zeta^*(s) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \varphi(h)\Theta_\psi^{\Phi_1}(h)E^{\Phi_2}(h,s)dh$$

To complete the proof, we note that $\zeta^*(s)$ is precisely the kind of Shimura-type zeta integral introduced in [GePS] to interpolate the L-function $L(s, Sh_\psi(\sigma))$! Thus it is not surprising that we end up with the basic identity

$$\zeta^*(s) = \left(\prod_{v\in S} \zeta_v^*(s)\right) L_S(s, Sh_\psi(\sigma)).$$

Here $\zeta_v^*(s)$ is a local zeta-integral which does not vanish identically at $s = \frac{1}{2}$ *if and only if* σ_v has a ψ_v-Whittaker model. Thus the theorem follows.

Remarks concerning the statement and proof of Waldspurger's Theorem

(a) The idea of the proof of Waldspurger's theorem can be summarized in one sentence: in manipulating the expression for a theta-series lifting, one runs smack up against a (special value of a) particular zeta-integral; thus the non-vanishing of the lifting must indeed be related to the non-vanishing of a special value of the automorphic L-function interpolated by this zeta-integral. An attempt to generalize this phenomenon was made in 1982 by Rallis, who considered higher dimensional orthogonal groups in place of PGL_2, and computed the L^2-norm of the resulting lifting in terms of a (then) new kind of zeta-integral of Rankin–Selberg type; see [Ra.2]. It was precisely these calculations which eventually gave rise to the general Rankin–Selberg–Godement–Jacquet zeta-integrals used by Piatetski-Shapiro and Rallis in their treatment of the standard automorphic L-functions attached to the simple classical groups (Theorem [P-R.1] of §II.1). It is therefore clear that the theory of theta-series liftings is inextricably linked with the theory of automorphic L-functions.

(b) Let us call a cuspidal representation ψ – *globally generic* if (ψ is a non-trivial character of the maximal unipotent homogeneous space $U(F)\backslash U(\mathbb{A}$ and) the space of ψth Fourier coefficients of cusp forms in the space of σ does not vanish identically. According to the statement of Waldspurger's theorem, σ need not be globally ψ-generic even though its factors σ_v are locally ψ_v-generic. In fact, explicit examples have been given (by Gelbart and Soudry) of cuspidal representations of $\overline{SL}_2(\mathbb{A})$ which are everywhere locally ψ_v-generic and hence possess 'abstract' ψ-Whittaker models globally, but are not globally ψ-generic. Clearly it will be interesting to resolve the following:

Problem. If π is a cuspidal representation of an *algebraic* reductive group G, and each π_v is ψ_v-generic, prove that π is globally ψ-generic.

A special instance of this problem for GSp_2 arises in the work of [Bl-Ra]. Some progress towards an affirmative solution has recently been made in the work of [Ku-Ra-So]; here $G = Sp_2$, and it is shown that local ψ-generic implies global ψ-generic *provided* the degree five L-function of π is non-vanishing at $Re(s) = 1$.

(c) The statement (and proof) of Waldspurger's theorem is not exactly true as stated since a few special cusp forms on the metaplectic group fail to lift to *cusp* forms on PGL_2. Examples of such cusp forms include the 'elementary theta-functions' on $\overline{SL}_2$ arising from the dual pair $(SL_2, O(1))$. The explanation for this phenomenon is a part of Rallis' theory of 'towers of Θ-series liftings', which we now briefly describe in this special context.

Consider the following sequence of orthogonal groups paired dually with SL_2:

$$\begin{array}{ccc} & & SO(3,2) \\ & \nearrow & \\ SL_2 & \to & SO(2,1) \\ & \searrow & \\ & & O(1) \end{array}$$

For each $j = 0, 1, 2$, let I_j denote the subspace of (genuine) cusp forms on $\overline{SL}_2(\mathbb{A})$ whose Θ-series lifts to $O(n+1, n)$ are zero for $n < j$, but not for $n = j$. According to [Ra.3]:

(i) $I_0 \oplus I_1 \oplus I_2$ *exhausts* the space of genuine cusp forms on $\overline{SL}_2$, and
(ii) if $\sigma \subset I_j$, then the theta-series lift of σ from $\overline{SL}_2$ to its dual pair partner $0(j+1, j)$ is automatically *cuspidal* (and non-zero); however, the theta-lift of this same σ to any 'larger' $O(n+1, n)$ is *non*-cuspidal (and non-zero).

In particular, for Waldspurger's dual pair $(\overline{SL}_2, PGL_2)$, the ψ theta-lift of σ will be non-cuspidal precisely when the theta-lift of this same σ to $O(1)$ is non-zero. Such cuspidal σ have non-vanishing ψ-Fourier coefficients and generalize the classical theta-series

$$\Theta_\chi(z) = \Sigma \chi(n) n^\nu e^{2\pi i n^2 z}$$

where $\nu = 0$ or 1, $\chi(-1) = (-1)^\nu$, and Θ_χ is a classical cusp form of weight $\frac{1}{2} + \nu$. It is precisely these cuspidal σ which spoil the Waldspurger bijection between certain cusp forms on $\overline{SL}_2$ and PGL_2, and hence must be removed. To sum up: the correct space of cusp forms for Waldspurger's bijective correspondence is precisely Rallis' space I_1.

What about the space I_2? It turns out that I_2 is just the space of genuine cusp forms on $\overline{SL}_2$ with vanishing ψ-Fourier coefficients. For example, if $\psi'(x) = \psi_t(x) = \psi(tx)$, with $t \notin (F^x)^2$, then the cusp forms on $\overline{SL}_2$ which are ψ'-theta lifts from $O(1)$ will have *vanishing* ψ-Fourier coefficients. (Classically, these lifts correspond to theta-series of the form $\Sigma\chi(n)n^\nu e^{2\pi i n^2 tz}$.) The space of all cusp forms in I_2 is the space which Piatetski-Shapiro isolated for study in [P-S.2] and showed to be of such great interest in connection with Ramanujan's conjecture. Indeed, let π on $SO(3,2) \approx PGSp_4$ be the ψ-theta-series lift of a cuspidal space σ in I_2. From Rallis' theory of towers, it follows that π is automatically cuspidal. What Piatetski-Shapiro proves in [P-S.2] is that each such π also 'satisfies' the following unusual properties:

(1) π provides a counterexample to the generalized Ramanujan conjecture; in fact, such π's contain the counterexamples of [Ku] if σ does not come from *any* theta-series attached to a quadratic form in 1-variable, and the counterexamples of [H-P] otherwise;

(2) π is a CAP (*cuspidal associated to a parabolic*) representation;

(3) the standard (degree 4) L-function of π is *not* entire, and in fact has a pole to the right of the line $Re(s) = 1$;

(4) π is not globally ψ-generic for any ψ; and

(5) π has a 'unipotent' component in the sense of §I.3.

It is this last property which seems to be at the root of the problem of extending Ramanujan's conjecture to groups beyond the context of $GL(n)$ (where properties (2)–(5) are never satisfied by cuspidal representations, and hence one still believes in the truth of the Conjecture). Note that the theory of towers works in the opposite direction as well. For example, corresponding to the diagram

$$\begin{array}{ccc} O(2) & \rightarrow & Sp_2 \\ & \searrow & \\ & & Sp_1 = SL_2 \end{array}$$

one concludes that 'cusp forms' on $O(2)$ which lift to zero on Sp_1, i.e., do not play a role in the Shalika–Tanaka construction of cusp forms on SL_2 from $O(2)$, are precisely the forms which lift to (non-zero) *cusp* forms on Sp_2. (It is crucial now that we deal with $O(2)$ in place of $SO(2)$.) Locally, at a place where the quadratic form is anisotropic, we saw earlier that there is just one representation of O_2 missing from the pairing with SL_2, namely the non-trivial character which is *trivial* on $SO(2)$. The resulting lifted cusp forms on Sp_2 are precisely the [H-P] counterexamples to Ramanujan's conjecture mentioned above, and in §3 of Part I.

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